1 Overview

Main ideas:

1. LU factorization, its use for solving several related matrix-vector equations
2. algorithm for LU factorization
3. application to circuit design (omit)

Examples in text:

1. use an LU factorization to solve a matrix-vector equation
2. find an LU factorization
3. compute transfer matrix of ladder network, design ladder network with given transfer matrix (omit)

2 Discussion and Worked Examples

2.1 The LU Factorization

Suppose we wish to solve several matrix-vector equations:

$$Ax = b_1\,, \quad Ax = b_2\,, \quad \ldots\,, \quad Ax = b_p$$

where A is a given matrix, say $m \times n$, and $b_1, \ldots, b_p$ are given vectors in $\mathbb{R}^m$. If A is square and invertible, we can find A^{-1} using row reduction techniques, and the solutions are $A^{-1}b_1$, $A^{-1}b_2$, $\ldots A^{-1}b_p$, respectively. In some sense, we have solved all the equations simultaneously. We would like a similar technique for matrices that are not square.

We write A as a produce of two matrices, L and U, where L is square, lower triangular, with ones on the diagonal, and U is a row echelon form of A.

When we row reduce A to echelon form, we keep track of our elementary row operations (elementary matrices!).

$$A \;\to\; E_1A \;\to\; E_2E_1A \;\to\; \ldots \;\to\; E_r \ldots E_2E_1A \;=\; U$$

Since elementary matrices are invertible, the product of elementary matrices is invertible, and

$$A \;=\; (E_r \ldots E_2E_1)^{-1}\,U$$

If all our row operations are row replacement operations of the form: add a multiple of an upper row to a lower row, then all the corresponding elementary matrices (and their inverses) are square lower triangular. The product of lower triangular matrices is also lower triangular, so the matrix $(E_r \ldots E_2E_1)^{-1}$ is lower triangular. Denoting this matrix by L, we have $A = LU$. If we have kept track of our elementary row operations, L will be the matrix obtained by performing the inverse row operations to the identity matrix, in reverse order, since

$$L \;=\; E_1^{-1}\,E_2^{-1}\,\ldots\,E_r^{-1}$$

Example Find the LU factorization of $A \;=\; \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$.

We row reduce A to echelon form (not reduced echelon form) using only row replacement operations, replacing lower rows with upper rows, and keeping track of each operation.

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \overset{R_1+R_2}{\to} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{bmatrix} \overset{-2R_1+R_3}{\to} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \overset{5R_2+R_3}{\to} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

This last matrix is our U. To find L, we apply the inverse row operations to the 3×3 identity matrix, in the reverse order:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{-5R_2+R_3}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \overset{2R_1+R_3}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \overset{-R_1+R_2}{\to} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

This matrix is our L. (Check that $LU = A$?)

Pattern The first column of L is the first column of A rescaled to start with a one (so, scale by $\frac{1}{3}$).

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}$$

The second column of L must start with a zero (in order for L to be lower triangular) and the entries below the zero are the (rescaled) entries below the pivot in the second column of

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix}$$

which is the matrix that occurs in row reduction just after the first column has been cleared. The entries are scaled by $-\frac{1}{2}$ in order to yield a one for the first nonzero entry of the second column of U:

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

The third column of L starts with two zeros and then (well, this is trivial, but) is followed by a 1, which can be thought of as a rescaled -1, which is the third entry of the third column of

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

which is the matrix occurring in the row reduction, just after the second column has been cleared. This works in general.

Example Find the LU factorization of $A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix}$. Again, we row reduce A to row echelon form.

This time, though, we will construct L simultaneously, using the observations above.

$$A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & -4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Next we will discuss the problem mentioned at the beginning: how to use an LU factorization to solve matrix-vector equations efficiently.

$$Ax = b \Leftrightarrow (LU)x = b \Leftrightarrow L(Ux) = b \Leftrightarrow Ly = b \quad \text{where } y = Ux$$

Since L is invertible, we can always find y such that $Ly = b$. We can do this using row reduction of the corresponding augmented matrix, and since L is lower triangular, row reduction is very quick. To find x, we solve $Ux = y$, if possible.

Example Solve $Ax = b$, where

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

As we found in the first example,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

First we solve $Ly = b$ by row reducing the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{bmatrix} \overset{R_1+R2}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 2 & -5 & 1 & 2 \end{bmatrix} \overset{-2R_1+R_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & 16 \end{bmatrix} \overset{5R_2+R_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Thus $y = (-7, -2, 6)$. Now we solve $Ux = y$, again by row reducing the corresponding augmented matrix:

$$\begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{bmatrix} \overset{-R_3}{\rightarrow} \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & -6 \end{bmatrix} \overset{R_3+R_2}{\rightarrow} \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

$$\overset{2R_3+R_1}{\rightarrow} \begin{bmatrix} 3 & -7 & 0 & -19 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{bmatrix} \overset{(-1/2)R_2}{\rightarrow} \begin{bmatrix} 3 & -7 & 0 & -19 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \overset{7R_2+R_1}{\rightarrow} \begin{bmatrix} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \overset{(1/3)R_1}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

Thus $x = (3, 4, -6)$. (Check that $Ax = b$?)

Example Solve $Ax = b$, where

$$A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

Note that this matrix is the same matrix as the second example, so

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix}$$

Again, we solve $Ly = b$ and then $Ux = y$. Start by row reducing the augmented matrix for $Ly = b$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & -4 \\ 0 & 1 & 1 & 6 \end{bmatrix} \overset{2R_1+R_2}{\to} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 6 \end{bmatrix} \overset{-R_2+R_3}{\to} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Thus $y = (2, 0, 6)$. Now we row reduce the augmented matrix for $Ux = y$:

$$\begin{bmatrix} 2 & -6 & 4 & 2 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & -2 & 6 \end{bmatrix} \overset{(-1/2)R_3}{\to} \begin{bmatrix} 2 & -6 & 4 & 2 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \overset{-8R_3+R_2}{\to} \begin{bmatrix} 2 & -6 & 4 & 2 \\ 0 & -4 & 0 & 24 \\ 0 & 0 & 1 & -3 \end{bmatrix} \overset{-4R_3+R_1}{\to} \begin{bmatrix} 2 & -6 & 0 & 14 \\ 0 & -4 & 0 & 24 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\overset{(-1/4)R_2}{\to} \begin{bmatrix} 2 & -6 & 0 & 14 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{bmatrix} \overset{6R_2+R_1}{\to} \begin{bmatrix} 2 & 0 & 0 & -22 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{bmatrix} \overset{(-1/2)R_1}{\to} \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Thus $x = (-11, -6, -3)$.

Note In exercises 24, 25, and 26 on the homework, you will work with three other important matrix factorizations: the QR factorization, the singular value decomposition, and the spectral factorization. We will discuss these factorizations and their uses in more detail later in the semester.

2.2 Other Matrix Factorizations

In exercises 24-26, you will explore three other matrix factorizations:

1. the QR factorization
2. the singular value decomposition
3. the spectral factorization

The goal of these exercises is for you to discover the usefulness of these factorizations. For example, in #24, ask yourself, "If I have the QR factorization for a matrix A, what question is now easier to answer?"