1 Overview

Main ideas in 2.2:

1. terminology: invertible matrix, inverse of a matrix, singular matrix, nonsingular matrix
2. criterion for invertibility of 2×2 matrix, formulae for inverse and determinant of 2×2 matrix
3. perspective on existence and uniqueness question
4. properties of inverses
5. elementary matrices, characterization of invertible $n \times n$ matrices, algorithm for finding inverse of $n \times n$ matrix

Main ideas in 2.3:

1. invertible matrix theorem
2. invertible linear transformations

Examples in 2.2:

1. verify inverse of 2×2 matrix
2. find inverse of 2×2 matrix
3. application: flexibility matrix, stiffness matrix
4. use inverse matrix to solve linear system
5. products with elementary matrices
6. inverse of elementary matrix
7. algorithm for finding inverse matrix

Examples in 2.3

1. determine if a matrix is invertible
2. characterize one-to-one linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$

2 Discussion and Worked Examples

Inverse of a 2×2 matrix:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is invertible,} \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the determinant of A and is denoted $\det A$. A matrix A is invertible if and only if $\det A \neq 0$.

Example Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and verify that $A\,A^{-1} = A^{-1}\,A = \mathbb{I}_2$.

According to the formula, $A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$. I will leave the matrix multiplication to you.

(Recap) existence and uniqueness questions from the four perspectives:

1. system of m linear equations in n unknowns, augmented matrix as shorthand
2. linear combinations of n vectors in $\mathbb{R}^m$
3. matrix-vector equation ($m \times n$ matrix)
4. linear transformations ($\mathbb{R}^m \to \mathbb{R}^n$)

Four equivalences when $n = m$.

1. all rows of coefficient matrix are pivot rows $\Leftrightarrow$ all columns of coefficient matrix are pivot columns
2. any vector in $\mathbb{R}^n$ can be written as a linear combination of specified set of n vectors $\Leftrightarrow$ the specified set of vectors is linearly independent
3. the columns of the matrix span $\mathbb{R}^n$ $\Leftrightarrow$ the homogeneous equation $Ax = b$ has no nontrivial solutions
4. linear transformation is one-to-one $\Leftrightarrow$ linear transformation is onto

From perspective (1) it is clear that we have another equivalent condition: the (coefficient) matrix is row equivalent to the identity matrix.

Further, if an $n \times n$ matrix A is invertible, then, for any b in $\mathbb{R}^n$, the matrix-vector equation $Ax = b$ has a unique solution, namely $x = A^{-1}b$. Thus

$$A \text{ is invertible} \quad \Rightarrow \quad Ax = b \text{ has a unique solution for all } b \text{ in } \mathbb{R}^n \quad \Rightarrow \quad A \text{ is row equivalent to } \mathbb{R}^n$$

Elementary Matrices Activity

We notice: row reduction can be acheived by repeatedly applying elementary matrices. If A is an $n \times n$ matrix and $R_A = E_r \dots E_1 A$ is the reduced row echelon form of A, either $R_A = \mathbb{I}_n$ or R_A has an identity block in the upper left corner and zeros all below. If R_A is the identity matrix, then clearly $E_r \dots E_1$ is A^{-1}. (This gives us an algorithm for computing a matrix inverse!) On the other hand, if $R_A \neq \mathbb{I}_n$, then A is not invertible. Thus

$$A \text{ is row equivalent to } \mathbb{I}_n \;\Rightarrow\; A \text{ is invertible}$$

Combined with the previous observations, we can conclude:

$$A \text{ is invertible} \quad \Leftrightarrow \quad \begin{array}{l} Ax = b \text{ has a unique solution for all } b \text{ in } \mathbb{R}^n \\ \text{(and all translations of this statement in} \\ \text{the other three perspectives)} \end{array} \quad \Leftrightarrow \quad A \text{ is row equivalent to } \mathbb{I}_n$$

Example Find the inverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & -4 & 0 \\ 6 & 5 & -3 \end{bmatrix}$.

We row reduce A and apply the same actions to the identity matrix:

$$\begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ -3 & -4 & 0 & 0 & 1 & 0 \\ 6 & 5 & -3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{add } R_1 \text{ to } R_2} \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 1 & 1 & 0 \\ 6 & 5 & -3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{add } (-2)R_1 \text{ to } R_3} \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 1 & 1 & 0 \\ 0 & 5 & -3 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{add } R_3 \text{ to } R_2} \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 1 \\ 0 & 5 & -3 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{add } (-5)R_2 \text{ to } R_3} \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 1 \\ 0 & 0 & 12 & 3 & -5 & -4 \end{bmatrix}$$

$$\xrightarrow{(1/12)R_3} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{5}{12} & -\frac{1}{3} \end{array}\right] \xrightarrow{\text{add } 3R_3 \text{ to } R_2} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{5}{12} & -\frac{1}{3} \end{array}\right] \xrightarrow{(1/3)R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{5}{12} & -\frac{1}{3} \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{5}{12} & -\frac{1}{3} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 4 & 0 & 0 \\ -3 & -3 & 0 \\ 3 & -5 & -4 \end{bmatrix}$$

Two more equivalent conditions:

A is invertible $\Leftrightarrow$ A^T is invertible

A is invertible $\Leftrightarrow$ A has a left inverse $\Leftrightarrow$ A has a right inverse