1 Overview

Main ideas:

1. def'n of null space of matrix (implicit description); null space of an $m \times n$ matrix is a ssp of $\mathbb{R}^n$; using row reduction to find explicit description of null space
2. def'n of column space of a matrix; column space of an $m \times n$ matrix is a ssp of $\mathbb{R}^m$; relation to systems of equations
3. def'n of (abstract) linear transformation, kernel and range of linear transformation

Examples in text:

1. determine whether given vector belongs to the null space of a given matrix
2. show that a set of vectors is a ssp of $\mathbb{R}^4$, by showing that it is the solution set of a homogeneous equation
3. find spanning set for the null space of given matrix
4. find a matrix such that the given subspace is the column space
5. for given matrix, determine k, ℓ such that column space is ssp of $\mathbb{R}^k$ and null space is ssp of $\mathbb{R}^\ell$
6. find nonzero vector in the column space and a nonzero vector in the null space
7. determine whether given vectors are in column space or in null space
8. kernel and range of transformation $f \to f'$ (differentiation)
9. kernel and range of $f \to f'' + \omega^2 f$

2 Discussion and Worked Examples

2.1 The Null Space of a Matrix

Definition The null space of an $m \times n$ matrix A is the set of all vectors in $\mathbb{R}^n$ such that $Ax = 0$.

Note This is a new name for an old friend. The null space of a matrix A is the solution set to the corresponding homogeneous linear system. It is also the kernel of the corresponding linear transformation.

Example Determine whether $(-6, 2, 2)$ is in the null space of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix}$.

We simply check with matrix multiplication:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, yes, v is in the null space of A.

Claim The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Proof:

Clearly the null space of A is a subset of the domain of the transformation $x \to Ax$, which is $\mathbb{R}^n$, so to show that it is a subspace, we need to show the following three things:

1. The zero vector is in the null space.
2. The sum of any two vectors in the null space is also in the null space.
3. Any scalar multiple of a vector in the null space is also in the null space.

Clearly the zero vector is in the null space because the product of any matrix and the zero vector of the appropriate size is the zero vector.

Now let v and w be vectors in the null space of A, and let c be any scalar. By linearity of the map $x \to Ax$,

$$\begin{aligned} A(v+w) &= Av + Aw = 0+0 = 0 \\ A(cv) &= c(Av) = c(0) = 0 \end{aligned}$$

Thus the null space is closed under addition and scalar multiplication, completing the proof that the null space is a subspace of $\mathbb{R}^n$.

Notice that the definition of the null space of a matrix does not provide an explicit description of the null space. It is relatively easy to check whether a given vector is in the null space, but this is not the same as having a description of the entire null space. To get an explicit description, we solve the homogeneous equation $Ax = 0$, and write the solution set in parametric vector form. A suitable spanning set is then obtained immediately from parametric vector form.

Example Give an explicit description of the null space of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & -3 \end{bmatrix}$$

We row reduce the augmented matrix for the equation $Ax = 0$:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 0 \\ -1 & 1 & 2 & 5 & 4 & 0 \\ 2 & -1 & -3 & 1 & 2 & 0 \\ 3 & 4 & 1 & 1 & -3 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = x_3$, $x_2 = -x_3 + x_5$, $x_4 = -x_5$, and x_3 and x_5 are free, i.e.

$$x = \begin{bmatrix} x_3 \\ -x_3 + x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

In other words, the null space consists of vectors of the form:

$$v = t \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad s, t \text{ in } \mathbb{R}$$

i.e. the null space is spanned by $\{(1,-1,1,0,0), (0,1,0,-1,1)\}$.

Note This spanning set is linearly independent, and the number of vectors in the spanning set is equal to the number of free variables. This is true generally when we obtain a spanning set for the null space by row reducing the matrix.

2.2 The Column Space of a Matrix

Definition The column space of an $m \times n$ matrix is the set of all vectors in $\mathbb{R}^m$ that can be written as a linear combination of the columns of A.

Note Again, this is just a new name for an old friend. The span of the columns of a matrix is the range (or image) of the associated linear transformation.

Since the column space of a matrix is defined as a span of a set of vectors in $\mathbb{R}^m$, it is a subspace of $\mathbb{R}^m$. In your homework you will prove this directly (i.e. by verifying that it satisfies the definition of a subspace.)

Note that, in contrast to the null space, the matrix itself provides an explicit description of the column space, but row reduction is required to check whether a given vector belongs to the column space.

Example Let $A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$.

1. The null space is a subspace of $\mathbb{R}^k$, for what k? ($k = 2$)
2. The column space is a subspace of $\mathbb{R}^\ell$, for what ℓ? ($\ell = 4$).
3. Determine whether the vector $(1, 0)$ is in the null space.

   To do this, we simply multiply by A:

   $$\begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -9 \\ 9 \end{bmatrix}$$

   Since this is nonzero, $(1, 0)$ is not in the null space.
4. Find a spanning set for the null space.

   To do this we row reduce the augmented matrix for the homogeneous equation $Ax = 0$:

   $$\begin{bmatrix} 6 & -4 & 0 \\ -3 & 2 & 0 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Thus $x_1 = (2/3)x_2$, and x_2 is free. So the solution set consists of vectors of the form

   $$v = t \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \qquad \text{for } t \text{ in } \mathbb{R}$$

   and the null space is spanned by $\{(2, 3)\}$ in $\mathbb{R}^2$.
5. Determine whether or not $(1, 0, 0, 0)$ is in the column space.

   To do this we row reduce the augmented matrix for the nonhomogeneous equation $Ax = (1, 0, 0, 0)$:

   $$\begin{bmatrix} 6 & -4 & 1 \\ -3 & 2 & 0 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Since the augmented matrix has a pivot in the last column, the associated linear system is inconsitent, i.e. there is no vector x such that $Ax = (1, 0, 0, 0)$. Thus the vector $(1, 0, 0, 0)$ cannot be written as a linear combinations of the columns of A, i.e. it does not lie in the column space.

6. Find a vector that is in the column space.

   Any vector that is a linear combination of the columns of A is in the column space. Notice that in this case the second column of A is a scalar multiple of the first. Thus the only vectors in the column space are scalar multiples of the first column. For example, $(-2, 1, 3, -3)$ is in the column space.

Example Let $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 9 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$.

1. The null space is a subspace of $\mathbb{R}^k$, for what k? $(k = 5)$
2. The column space is a subspace of $\mathbb{R}^\ell$, for what ℓ? $(\ell = 2)$.
3. Find a spanning set for the null space.

   To do this we row reduce the augmented matrix for the homogeneous equation $Ax = 0$:

$$\begin{bmatrix} 4 & 5 & -2 & 6 & 9 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \ \dots \ \rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{bmatrix}$$

   Thus $x_1 = -2x_3 + x_4$, $x_2 = 2x_3 - 2x_4$, and x_3, x_4, and x_5 are free. So the solution set consists of vectors of the form

$$v = \begin{bmatrix} -2t + s \\ 2t - 2s \\ t \\ s \\ r \end{bmatrix} = t \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{for } t, r, s \text{ in } \mathbb{R}$$

   and the null space is spanned by $\{(-2, 2, 1, 0, 0), (1, -2, 0, 1, 0), (0, 0, 0, 0, 1)\}$ in $\mathbb{R}^5$.

4. Determine whether or not $(1, 0)$ is in the column space.

   In general, we would need to row reduce the augmented matrix for the nonhomogeneous equation $Ax = (1, 0)$. However, in this case, we can answer the question without row reduction. Notice that the first two columns of A are linearly independent. Thus the columns of A span $\mathbb{R}^2$. Thus any vector in $\mathbb{R}^2$ is in the column space of A, and, in particular, $(1, 0)$ is.

2.3 Linear Transformations

Definition A function T from a vector space V to a vector space W is a linear transformation if it respects vector addition and scalar multiplication, i.e.

$$T(u + v) = T(u) + T(v) \qquad \text{and} \qquad T(cv) = c\,T(v) \qquad \text{for all vectors } u, v \text{ in } V \text{ and all scalars } c$$

The kernel of T is the set of all v in V such that $T(v) = 0$, and the range (or image) of T is the set of all w in W of the form $T(v)$ for some v in V.

Example Let V be the set of all continuous functions on $[0, 2\pi]$. Then integration over $[0, 1]$ is a linear transformation from V to $\mathbb{R}$:

$$T(f) = \int_0^{2\pi} f(x)\,dx$$

What is the kernel of T? Can you give an example of a function in the kernel? What is the image?

The kernel of T is the set of all continuous functions on $[0, 2\pi]$ whose integral over $[0, 2\pi]$ is zero:

$$f(x) \text{ is in } \ker(T) \iff \int_0^{2\pi} f(x)\,dx = 0$$

For example $f(x) = \sin(x)$ is in the kernel. A real number A is in the range of T if there is a continous function $f(x)$ on $[0, 2\pi]$ such that

$$\int_0^{2\pi} f(x)\,dx \;=\; A$$

Given a real number A, the constant function $f_A(x) \;=\; A/2\pi$ satisfies the desired property; thus the range of T is all of $\mathbb{R}$.