1 Overview

Main ideas:

1. definition of symmetric matrix
2. Spectral Theorem: orthogonal diagonalization of symmetric matrices
3. spectral decomposition of a symmetric matrix

Examples in text:

1. examples of symmetric and non-symmetric matrices
2. diagonalizing a symmetric matrix; notice it is orthogonally diagonalizable
3. diagonalize a symmetric matrix, using the fact that it is orthogonally diagonalizable
4. spectrally decompose a matrix, given orthogonal diagonalization

2 Discussion and Worked Examples

2.1 Diagonalizing a symmetric matrix

Recall that a square matrix A is *diagonalizable* if there is an invertible matrix P and a (not necessarily invertible) diagonal matrix D such that $A = PDP^{-1}$.

More specifically, an $n \times n$ matrix A is diagonalizable if and only if its characteristic polynomial factors into linear factors, and each eigenspace has dimension equal to the multiplicity of the corresponding eigenvalue. In this case, the sum of the dimensions of its eigenspaces is n, so there is a basis for $\mathbb{R}^n$ consisting of eigenvectors for A. We form the matrix P by letting its columns be the basis of eigenvectors. The entries of the matrix D are the corresponding eigenvalues (in order).

To find the eigenvalues of a matrix A, find the roots of the characteristic equation $\det(A - \lambda\mathbb{I}) = 0$. The vectors v in $\mathbb{R}^n$ satisfying $Av = \lambda v$ form the eigenspace V_λ, and $V_\lambda = \text{null}(A - \lambda\mathbb{I})$. Thus row reduction of $(A - \lambda\mathbb{I})$ can be used to find a basis for V_λ.

Activity: Diagonalizing a symmetric matrix.

Consider the symmetric matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1. Compute the eigenvalues of A, and, for each distinct eigenvalue λ, find a basis for the corresponding eigenspace V_λ.
2. For each pair λ and μ of distinct eigenvalues, compute the dot products of the basis vectors for V_λ with the basis vectors for V_μ. What do you notice?
3. Normalize each eigenvector to be a unit vector, and let Q be the matrix whose columns are the unit eigenvectors. Given your observation from (2), what kind of matrix is Q?
4. Given (3), how can you find Q^{-1} quickly (e.g. row reduction)? What are the implications for diagonalizing A?

The matrix A in this example is called *symmetric* because $A^T = A$. We have seen that it is possible to diagonalize A by orthogonal matrices. Such a matrix is called *orthogonally diagonalizable.* It turns out that all symmetric matrices are orthogonally diagonalizable and no other matrices are orthogonally diagonalizable.

Theorem (Spectral Theorem and Converse)**.** *Let A be an $n \times n$ symmetric matrix. Then*

1. *A has n real eigenvalues, counting multiplicities*
2. *The dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue.*
3. *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*

i.e. A is orthogonally diagonalizable. Conversely, any orthogonally diagonalizable matrix is symmetric.

Proof. We prove only two parts of this theorem: first, that eigenvectors corresponding to distinct eigenvalues are orthogonal, and second, that an orthogonally diagonalizable matrix is symmetric.

Let A be a symmetric matrix with distinct eigenvalues λ and μ and corresponding eigenvectors v and w, respectively. Then

$$Av \cdot w \;=\; v^T A^T w \;=\; v^T A w \;=\; v \cdot Aw$$

Thus $\lambda\, v \cdot w \;=\; \mu\, v \cdot w$. Since $\lambda \neq \mu$, we can conclude that $v \cdot w = 0$.

Suppose B is an orthogonally diagonalizable matrix. Then there is an orthogonal matrix Q and a diagonalizable matrix D such that $B = QDQ^T$. Then

$$B^T \;=\; \left(QDQ^T\right)^T \;=\; \left(Q^T\right)^T D^T Q^T \;=\; QDQ^T \;=\; B$$

Thus B is symmetric. □

2.2 The Spectral Decomposition

Recall that one interpretation of diagonalizing a matrix is finding a basis with respect to which the linear transformation acts by scalars. If a matrix A is diagonalizable, we can decompose $\mathbb{R}^n$ as the sum of the eigenspaces. On each eigenspace A acts like a dilation by the corresponding eigenvalue.

When the eigenspaces are mutually orthogonal, this gives an orthogonal decomposition of $\mathbb{R}^n$. Thus the linear transformation can be written as a sum of linear transformations, each one of which can be described as a projection onto an eigenvector followed by a scaling by the corresponding eigenvalue. In particular, if $\{u_1, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for a matrix A and if $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, then

$$Av \;=\; \lambda_1 \mathrm{proj}_{u_1} v \;+\; \ldots \;+\; \lambda_n \mathrm{proj}_{u_n} v$$

Notice that

$$\mathrm{proj}_{u_1} v \;=\; (v \cdot u_1)u \;=\; (v^T u_1)u_1 \;=\; (u_1^T v)u_1 \;=\; u_1(u_1^T v) \;=\; (u_1 u_1^T)v$$

Thus the map $v \mapsto \mathrm{proj}_{u_1} v$ has a matrix representation $u_1 u_1^T$. This gives the spectral decomposition of A:

$$A \;=\; \lambda_1(u_1 u_1^T) \;+\; \ldots \;+\; \lambda_n(u_n u_n^T)$$

Example For the matrix A in the activity above, the spectral decomposition is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$