1 Overview

Main ideas:

1. effect of row operations on the determinant of a matrix, using row reduction to compute determinants efficiently, factoring out common multiple of a row
2. a square matrix is invertible if and only if its determinant is nonzero
3. the determinant of a matrix is the same as the determinant of its transpose
4. the determinant of a product is the product of the determinants

Examples in text:

1. compute determinant after row reducing to echelon form
2. factoring out common multiple of a row to make computation of determinant simpler
3. determine that a determinant is zero by observing that columns (or rows) are linearly dependent
4. use row operations to introduce zeros before performing a cofactor expansion
5. verify that the determinant of a product of 2×2 matrices is the product of the determinants

2 Discussion and Worked Examples

Recall from the exercises that you did for homework:

- row interchanges multiply the determinant by -1
- row scaling multiplies the determinant by the scaling factor
- row replacements do not change the determinant

Determinants of elementary matrices:

$$\begin{vmatrix} 1 & & & & \\ & & 1 & & \\ & 1 & & & \\ & & & 1 & \\ & & & & 1 \end{vmatrix} = -1 \qquad \begin{vmatrix} 1 & & & & \\ & a & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{vmatrix} = a \qquad \begin{vmatrix} 1 & & & & \\ & 1 & & * & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{vmatrix} = 1$$

Thus if A is a square matrix and E is an elementary matrix of the same size, $\det(EA) = (\det E)(\det A)$. Using row reduction, we can write a matrix A as a product $E_r \dots E_1 U$, where $E_1, \dots, E_r$ are elementary matrices and U is an echelon form of A. Clearly the determinant of $E_r \dots E_1$ is nonzero (because it is a product of ± 1s and nonzero scalars). Thus $\det A = 0$ if and only if $\det U = 0$. Now if A is invertible, then U is upper triangular, thus has nonzero determinant. On the other hand, if A is not invertible, then U has a row of zeros, and its determinant is zero. Thus

A square matrix is invertible if and only if its determinant is nonzero.

This result is not surprizing–we knew it to be the case for 2×2 matrices–but perhaps we can all sleep better tonight knowing that the funny combinatorial definition of the determinant of an $n \times n$ matrix is in fact a necessary and sufficient condition for invertibility.

An easy corollary of this discussion is the fact that

The determinant of a product is the product of the determinants.

Suppose A and B are square matrices of the same size. If one of them is not invertible, then their product is not invertible, thus the product of their determinants is zero and so is the determinant of their product. If, on the other hand, both are invertible, then each can be written as the product of elementary matrices, say,

$$A = E_1 \dots E_r \quad \text{and} \quad B = E'_1 \dots E'_p$$

and thus $\det(AB)$ is

$$\det(E_1 \dots E_r\, E'_1 \dots E'_p) = \det(E_1) \dots \det(E_r)\, \det(E'_1) \dots \det(E'_p) = \det(E_1 \dots E_r) \cdot \det(E'_1 \dots E'_p)$$

i.e. $\det(AB) = \det(A) \cdot \det(B)$.

One last observation before turning to computation.

The determinant of a matrix is the same as the determinant of its transpose.

This is clear for a 1×1 matrix or a 2×2 matrix. Since the determinants of larger matrices are computed via cofactor expansions (computing determinants of submatrices) it makes sense that this result should hold in general.

Consequences for computing determinants:

- can row reduce to echelon form (keeping track of the number of interchanges and all of the scaling operations)
- can factor out a common multiple of a row
- if possible, use row replacement to see that the rows or columns of a matrix are linearly dependent, and thus the determinant is zero
- can use row replacement to introduce more zeros into the matrix, without changing the determinant
- can also use column operations (equivalent to row operations on the transpose!)

Example Compute the determinant by row reduction to echelon form

$$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$$

We row reduce the matrix, keeping track of how the determinant changes for row-scaling and row-interchange:

$$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} = 6 \cdot \begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 6 \cdot \begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 6 \cdot (-3) = -18$$

Example Use determinants to find out if the matrix is invertible:

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

We use row replacement to introduce another zero in the last column, use row scaling to factor out a (-1) from the second row, and then do a cofactor expansion down the last column:

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 \\ -3 & -5 & 0 \\ 1 & 2 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = (-1)(10-9) = -1$$

Since the determinant is nonzero, the matrix is invertible.

Example Use determinants to decide if the set of vectors is linearly independent:

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

This set of three vectors in $\mathbb{R}^3$ is linearly independent if and only if the matrix formed by taking these vectors as columns has nonzero determinant.

$$\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -3 & 7 \\ -4 & 5 & 0 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 7 \\ 1 & 5 & 0 \\ 0 & -1 & 2 \end{vmatrix} = -(-6+7) = -1$$

Thus the set of vectors is linearly independent.