1 Overview

Main ideas:

1. quadratic form, matrix of quadratic form, cross-product terms, change of variables

2. Principal Axes Theorem

3. positive definite, negative definite, indefinite quadratic forms; characterization using eigenvalues

Examples in text:

1. evaluate quadratic form on $\mathbb{R}^2$

2. find matrix of a quadratic form on $\mathbb{R}^3$

3. evaluate quadratic form on $\mathbb{R}^2$

4. change variables to eliminate cross-product term

5. change variables to eliminate cross-product term

6. determine whether a quadratic form on $\mathbb{R}^3$ is positive definite

2 Discussion and Worked Examples

2.1 Introduction to Quadratic Forms

A *quadratic form* on $\mathbb{R}^n$ is a function $Q : \mathbb{R}^n \to \mathbb{R}$ that can be represented as $Q(v) = v^T A v$ for an $n \times n$ symmetric matrix A.

Example Let $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$, and let Q be the quadratic form associated to A. Then

$$Q((6,1)) = \begin{bmatrix} 6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \end{bmatrix} \begin{bmatrix} 91/3 \\ 3 \end{bmatrix} = 182 + 3 = 185$$

$$Q((1,3)) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 10/3 \end{bmatrix} = 16$$

$$Q((x_1, x_2)) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5x_1 + x_2/3 \\ x_1/3 + x_2 \end{bmatrix} = x_1(5x_1 + x_2/3) + x_2(x_1/3 + x_2)$$

$$= 5x_1^2 + x_1 x_2/3 + x_1 x_2/3 + x_2^2 = 5x_1^2 + (2/3)x_1 x_2 + x_2^2$$

Here is a 3D plot of the quadratic form:
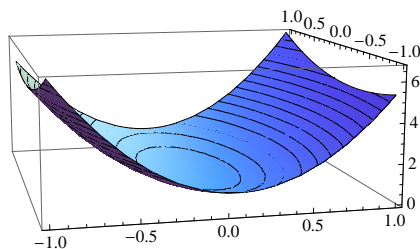
Example Let $A = \begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$. Find $Q\big((x_1, x_2, x_3)\big)$.

$$Q\big((x_1, x_2, x_3)\big) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5x_1 + 5x_2/2 - 3x_3/2 \\ 5x_1/2 - x_2 \\ -3x_1/2 + 7x_3 \end{bmatrix} = 5x_1^2 + 5x_1x_2 - 3x_1x_3 - x_2^2 + 7x_3^2$$

Example Find the matrix of the quadratic form $Q\big((x_1, x_2, x_3)\big) = x_3^2 - 4x_1x_2 + 4x_2x_3$.

The coefficients of x_1^2, x_2^2 and x_3^2 should be the diagonal entries.

$$\begin{bmatrix} 0 & ? & ? \\ ? & 0 & ? \\ ? & ? & 1 \end{bmatrix}$$

The coefficient of x_1x_2 should be twice the entry in the first row, second column (which is the same as the entry in the second row, first column.)

$$\begin{bmatrix} 0 & -2 & ? \\ -2 & 0 & ? \\ ? & ? & 1 \end{bmatrix}$$

Similarly the coefficient of the x_2x_3 term should be twice the entry in the second row, third column (which is the same as the entry in the third row, second column).

$$\begin{bmatrix} 0 & -2 & ? \\ -2 & 0 & 2 \\ ? & 2 & 1 \end{bmatrix}$$

Since the coefficient of the x_1x_3 term is zero, the remaining two entries (first row third column and third row first column) should be zeros. So the matrix for the quadratic form is

$$\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

2.2   Eliminating Cross-terms

A quadratic term without cross terms is of the form $Q(x) = a_1 x_1^2 + \ldots + a_n x_n^2$ and corresponds to a diagonal matrix. Given a quadratic form with cross-terms, we can eliminate the cross-terms by a change of variables obtained by orthogonally diagonalizing the corresponding matrix.

Example The quadratic form $Q(x) = 4x_1^2 - 2x_1x_2 + 4x_2^2$ has an associated matrix which can be orthogonally diagonalized as

$$A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} = PDP^T = \begin{bmatrix} 1/\sqrt{2} & -1\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Thus

$$Q(x) = x^T A x = x^T (PDP^T) x = (x^T P) D (P^T x) = (P^{-1}x)^T D (P^{-1}x)$$

If we let $y = P^{-1}x$, then we have

$$4x_1^2 - 2x_1x_2 + 4x_2^2 = 3y_1^2 + 5y_2^2$$

The columns of P are eigenvectors of A. Since A is symmetric, these eigenvectors form an orthonormal basis for $\mathbb{R}^2$. We can think of them as giving a new set of coordinate axes along which A acts by scaling. These are called the *principal axes* of the quadratic form Q.

Here is a 3D plot of the quadratic form and a contour plot with the principal axes marked.
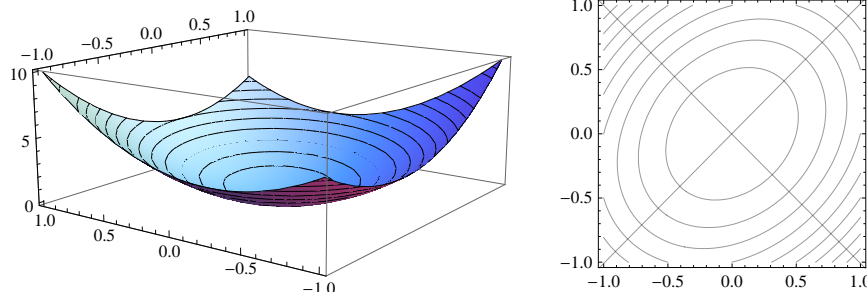
The **Principal Axes Theorem** says that we can always find a change of variables that transforms the quadratic form into a quadratic form with no cross terms. This follows from the fact that all symmetric matrices are orthogonally diagonalizable, as discussed the example above.

2.3   Classifying Quadratic Forms

Quadratic forms are classified as *positive definite*, *negative definite*, or *indefinite* according to whether they assume only positive values (for $x \neq 0$), only negative values (for $x \neq 0$), or both positive and negative values.

It turns out that it is sufficient to look at the *signs of the eigenvalues* to determine whether a quadratic form is positive definite, negative definite, or indefinite. This is because it is always possible to find principal axes and transform the quadratic form such that it has no cross-terms. Once the cross terms have been eliminated, the coefficients are precisely the eigenvalues of the original matrix.

Example The orthogonal diagonalization in the previous example

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

led to the change of variables

$$4x_1^2 - 2x_1 x_2 + 4x_2^2 = 3y_1^2 + 5y_2^2$$

Clearly the quadratic form is positive definite, since $y_1^2, y_2^2 \geq 0$ and the eigenvalues 3 and 5 are both positive.

Example An example of an indefinite quadratic form is $5x_1^2 + 2x_1 x_2 + 5x_2^2$. The associated matrix $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ has eigenvalues 6 and -4. Change-of-variables yields $6y_1^2 - 4y_2^2$, which can be positive or negative.