1 Overview

Main ideas:

1. row space of a matrix; row equivalent matrices have the same row space
2. rank of a matrix; rank theorem
3. application to systems of equations
4. rank of invertible matrices

Examples in text:

1. row space of a matrix
2. find bases for row sp, col. sp, and null sp of a matrix
3. dimensions of row, col, null sp
4. visualizing row sp and null sp
5. application to systems of equations

2 Discussion and Worked Examples

2.1 Row Space

Recall that the span of the columns, i.e. the column space, of a matrix is the image/range of the associated linear transformation and is a subspace of the codomain. Questions about existence of solutions of linear systems pertain to the column space of the matrix. (The vector b is in the column space of a matrix A if and only if the equation $Ax = b$ has a solution.)

Recall that the null space of a matrix is the kernel of the associated linear transformation and is a subpace of the domain. Questions about uniqueness of solutions of linear systems pertain to the null space of a matrix. (The equation $Ax = b$, if consistent, has a unique solution if and only if the null space of A is trivial.)

Recall that the dimension of the column space of a matrix is equal to the number of pivot columns and the dimension of the null space is equal to the number of columns without a pivot:

$$\dim(\mathrm{col}(A)) \;+\; \dim(\mathrm{null}(A)) \;\;=\;\; (\text{number of columns of } A)$$

It turns out that these subspaces are related to the corresponding subspaces of the transpose matrix. Notice that, if A is an $m \times n$ matrix, then the null space of A is a subspace of $\mathbb{R}^n$ and the column space of A is a subspace of $\mathbb{R}^m$, whereas the null space of A^T is a subspace of $\mathbb{R}^m$ and the column space of A^T is a subspace of $\mathbb{R}^n$. Because the columns of A^T are the rows of A, we often talk about the *row space* of A instead of the column space of A^T.

Key Observation: Performing row operations on a matrix does not change the span of the rows. (Interchanging two rows does not change the span of the rows. Scaling a row does not change the span of the rows. Replacing a row with a linear combination of that row and another does not change the span of the rows.)

So to find a basis for the row space of a matrix, row reduction will be helpful yet again. Once a matrix is in row echelon form, its rows are linearly independent, and so they form a basis for its row space and the row space of the original matrix.

2.2 Column Space, Row Space, and Null Space Together

Example Find bases for the row space, the column space, and the null space of the following matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & 1 \\ 1 & 2 & -3 & 0 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -2 & 2 & 1 & -3 & -2 \\ 1 & -2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

First we row reduce to echelon form, then to reduced echelon form.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 1 & 1 \\ 1 & 2 & -3 & 0 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -2 & 2 & 1 & -3 & -2 \\ 1 & -2 & 1 & 0 & 2 & 2 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & -3 & -3 \\ 0 & 0 & 1 & 1 & -13 & -12 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As we can see from echelon form all but the fourth and sixth columns are pivot columns, so the a basis for the column space of A is

$$\text{Basis for col}(A) \ : \ \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 2 \end{bmatrix} \right\}$$

The row space of A is the same as the row space of A_{ech} and the row space of A_{rref}. It is easy to see that the first four rows of A_{ech} are linearly independent. Since the row space of A and the row space of A_{ech} are the *same*, the first four rows of A_{ech} form a basis for the row space of A:

$$\text{Basis for row}(A) \ : \ \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -13 \\ -12 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To find a basis for the null space we need the reduced echelon form. We can see that a vector x in the null space will be of the form:

$$x = \begin{bmatrix} -x_4 - x_6 \\ -x_4 - x_6 \\ -x_4 - x_6 \\ x_4 \\ -x_6 \\ x_6 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad x_4, x_6 \text{ in } \mathbb{R}$$

Thus a basis for the null space of A is

$$\text{Basis for null}(A) \ : \ \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

In particular, notice that the rows in A_{ech} (or A_{rref}) with a pivot form a basis for the row space of A, so the dimension of row(A) is equal to the number of pivot positions of A, which is equal to dimension of the column space. Thus

$$\dim(\text{row}(A)) \quad = \quad \dim(\text{col}(A))$$

i.e. the dimension of the image of transformation $x \mapsto Ax$ is the same as the dimension as the image of $x \mapsto A^T x$. This number is called the *rank* of A.

Thus if A is an $m \times n$ matrix of rank r, the kernel of the associated linear transformation is an $(n-r)$-dimensional subspace of $\mathbb{R}^n$, and the image of the linear transformation associated to A^T is an r-dimensional subspace of $\mathbb{R}^n$. In the example above,

$$\text{Basis for col}(A^T)\ :\ \left\{ \begin{bmatrix} 1\\0\\0\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1\\1 \end{bmatrix} \right\} \qquad \text{Basis for null}(A)\ :\ \left\{ \begin{bmatrix} -1\\-1\\-1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} -1\\-1\\-1\\0\\-1\\1 \end{bmatrix} \right\}$$

It turns out that, together, these form a basis for $\mathbb{R}^6$. Thus we have two complementary subspaces of $\mathbb{R}^6$, in the sense that

$$\mathbb{R}^6 \quad = \quad \text{null}(A) \ + \ \text{col}(A^T)$$

and their intersection consists only of the zero vector.

2.3 Invertible Matrices

Recall that an $n \times n$ matrix A is invertible if and only if its columns span $\mathbb{R}^n$. Now we know that this is equivalent to: $\text{col}(A) = \mathbb{R}^n$ and to: $\text{rank}(A) = n$.

Recall that an $n \times n$ matrix A is invertible if and only if the homogeneous equation $Ax = 0$ has only the trivial solution. Now we can say that this is also equivalent to: $\text{null}(A) = \{0\}$ and to: $\dim(\text{null}(A)) = 0$.