1 Overview

Main ideas:

1. Terminology: nonzero row or column, leading entry of row, matrix in (row) echelon form, a matrix in reduced (row) echelon form, pivot position, pivot column
2. Uniqueness of reduced row echelon form
3. Row reduction algorithm (forward phase and backward phase)
4. Solving a linear system: basic variables, free variables, parametric description of solution set, back-substitution, determining existence and uniqueness from echelon form (not necessarily reduced)

Examples in the text:

1. Matrices in echelon form and reduced echelon form
2. Row reduce a matrix to echelon form and determine pivot columns
3. Row reduce a matrix to reduced echelon form
4. Find general solution to linear system whose augmented matrix has been reduced to given matrix.
5. Existence and uniqueness of solutions of a linear system.

2 Discussion and Worked Examples

Recall the linear system (and corresponding augmented matrix) that we considered last time:

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ -12x & + & 6y & + & 2z & = & -40 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ -12 & 6 & 2 & -40 \end{pmatrix}$$

Using elimination of variables, we found,

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ & & & & 5z & = & 20 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 5 & 20 \end{pmatrix}$$

It is apparent now that the linear system is consistent, because it is clear that $z = 4$, and we could use back substitution to solve for x and y. This matrix is in **(row) echelon form**. In general, a matrix is in row echelon form if

1. If there are any zero rows (all zeros), they are at the bottom.
2. The first nonzero entry of a row (called the leading entry) must be to the right of the leading entries of the rows above it.
3. There are all zeros below a leading entry.

Exercise Modify the example above to make two larger matrices, one of which is in row echelon form and the other is not.

$$\begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{is in row echelon form, but} \quad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 5 & 20 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{is not.}$$

Continuing with the example from last time, remember that instead of back-substituting, we continued eliminating variables until we had:

$$\begin{aligned} x \qquad\qquad &= 3 \\ y \qquad &= -2 \\ z &= 4 \end{aligned} \qquad\qquad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

This augmented matrix is in **reduced (row) echelon form**. In general, a matrix is in reduced row echelon form (rref) if, in addition to the three criteria above,

1. The leading entries are all ones.
2. All entries directly above and below the leading entry are zeros.

Exercise Modify the example above to make two larger matrices, one of which is in row echelon form and the other is not.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \quad \text{is in rref, but} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix} \quad \text{is not.}$$

Terminology: nonzero row or column, leading entry of row, matrix in (row) echelon form, a matrix in reduced (row) echelon form, pivot position, pivot column

Note: The reduced row echelon form of a matrix is *unique*. Why should this be so?

How to get a matrix into reduced row echelon form?

The idea is to start at the top left corner (first pivot), clear all entries below it using elementary row operations, then move down and to the right to find the next pivot and clear all entries below it, and so on until you've cleared all entries that lie below pivots. (This is the "forwards phase" of row reduction.) Now your matrix is in row echelon form (triangular).

To get it into reduced row echelon form, you start with the last pivot, (divide if necessary to get it to by one), clear all entries above it, and then move up and to the left to the previous pivot, divide if necessary and then clear all entries above it, and so on until you've scaled all pivots to be one and cleared all entries that lie above pivots. (This is called the "backwards phase.")

Here is an example. Try to finish it yourself instead of reading all the steps.

$$\begin{pmatrix} 4 & -2 & 1 & 20 \\ 1 & 1 & 1 & 5 \\ 9 & 3 & 1 & 25 \end{pmatrix} \xrightarrow{\text{mult } R_2 \text{ by } (-4)} \begin{pmatrix} 4 & -2 & 1 & 20 \\ -4 & -4 & -4 & -20 \\ 9 & 3 & 1 & 25 \end{pmatrix} \xrightarrow{\text{add } R_1 \text{ to } R_2} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & -6 & -3 & 0 \\ 9 & 3 & 1 & 25 \end{pmatrix}$$

$$\cdots \xrightarrow{\text{mult } R_3 \text{ by } (-4)} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & -6 & -3 & 0 \\ -36 & -12 & -4 & -100 \end{pmatrix} \xrightarrow{\text{add 9 times } R_1 \text{ to } R_3} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & -6 & -3 & 0 \\ 0 & -30 & 5 & 80 \end{pmatrix}$$

$$\ldots \xrightarrow{\text{mult } R_2 \text{ by } (-1)} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & -30 & 5 & 80 \end{pmatrix} \xrightarrow{\text{add 5 times } R_2 \text{ to } R_3} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 20 & 80 \end{pmatrix} \xrightarrow{\text{mult } R_3 \text{ by } (1/20)} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\ldots \xrightarrow{\text{add (-3) times } R_3 \text{ to } R_2} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 0 & -12 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{\text{mult } R_2 \text{ by } (1/6)} \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{\text{add (-1) times } R_3 \text{ to } R_1} \begin{pmatrix} 4 & -2 & 0 & 16 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\ldots \xrightarrow{\text{add 2 times } R_2 \text{ to } R_1} \begin{pmatrix} 4 & 0 & 0 & 12 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{\text{mult } R_1 \text{ by } (1/4)} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \quad \ldots \text{ and we're done!}$$

Note: There is more than one way to do this, but the last matrix you have should be exactly what I have.

Let's look at another example. Remember the traffic flow example from last time? We had the following system of equations:

$$\begin{aligned} x_1 + x_2 &= 800 \\ x_2 + x_4 &= x_3 + 300 \\ 500 &= x_4 + x_5 \\ x_1 + x_5 &= 600 \end{aligned}$$

Let's rewrite this to make the augmented matrix visible:

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & & & & & & & = & 800 \\ & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\ & & & & & & x_4 & + & x_5 & = & 500 \\ x_1 & & & & & & & + & x_5 & = & 600 \end{array} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{pmatrix}$$

Apply the row reduction algorithm:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \end{pmatrix} \xrightarrow{\text{add } (-1) \text{ times } R_1 \text{ to } R_4} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & -1 & 0 & 0 & 1 & -200 \end{pmatrix}$$

$$\ldots \xrightarrow{\text{add } R_2 \text{ to } R_4} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & -1 & 1 & 1 & 100 \end{pmatrix} \xrightarrow{\text{switch } R_3 \text{ and } R_4} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix}$$

$$\cdots \xrightarrow{\text{add } (-1) \text{ times } R_4 \text{ to } R_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 0 & 0 & -400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix} \xrightarrow{\text{mult } R_3 \text{ by } (-1)} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix}$$

$$\cdots \xrightarrow{\text{add } (-1) \text{ times } R_4 \text{ to } R_2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix} \xrightarrow{\text{add } R_3 \text{ to } R_2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix}$$

$$\cdots \xrightarrow{\text{add } (-1) \text{ times } R_2 \text{ to } R_1} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix} \quad \ldots \text{ reduced row echelon form!}$$

How to express the solution of the corresponding linear system?

In the first example, there is a unique solution:

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \quad \longleftrightarrow \quad \begin{array}{rcr} x & = & 3 \\ y & = & -2 \\ z & = & 4 \end{array}$$

However, in the second example, there is not a unique solution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix} \quad \longleftrightarrow \quad \begin{array}{rcr} x_1 + x_5 & = & 600 \\ x_2 - x_5 & = & 200 \\ x_3 & = & 400 \\ x_4 + x_5 & = & 500 \end{array}$$

Since there are four equations and five unknowns, we have a free variable. Last time we let $x_5 = A$, an undetermined constant, and we expressed the other variables ("basic variables") in terms of A.

$$x_1 = 600 - A, \quad x_2 = 200 + A, \quad x_3 = 400, \quad x_4 = 500 - A, \quad x_5 = A$$

This is called a parametric description of the solution set, because we are expressing the solution in terms of a free "parameter," namely A. In the book, they do not replace x_5 by another variable, as we did, instead they just say that x_5 is free:

$$x_1 = 600 - x_5, \quad x_2 = 200 + x_5, \quad x_3 = 400, \quad x_4 = 500 - x_5, \quad x_5 \text{ is free}$$

In any case, the point is that there is a whole *family* of solutions of the linear system and the family is parametrized by x_5, because any choice of x_5 gives a solution. Choosing $x_5 = 300$, would give us one solution, choosing $x_5 = \pi$ would given another, etc.