1 Overview

We describe solutions of linear systems in terms of vectors and interpret these descriptions geometrically. We first discuss the solution sets of "homogeneous" linear systems and then discuss how the solution set to any consistent linear system can be described in terms of the solution set of the corresponding homogeneous linear system.

Main ideas:

1. homogeneous linear systems: definition, solutions, trivial and nontrivial
2. solution set is always the span of a suitable set of vectors, "parametric vector form," geometric interpretation
3. solutions of nonhomogeneous systems, relation to solution set of corresponding homogeneous system, geometric interpretation

Examples in text:

1. solution set of a homogeneous linear system
2. homogeneous linear "system" of only one equation
3. solution set of a nonhomogeneous linear system

2 Discussion and Worked Examples

Example 1 Consider the following (homogeneous) linear system:

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 0 \\ 2x & + & y & - & 3z & = & 0 \\ -x & + & y & & & = & 0 \end{array}$$

To solve the linear system, we look at the corresponding augmented matrix and row reduce, as usual:

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add -2 } R_1 \text{ to } R_2} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add } R_1 \text{ to } R_3} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \xrightarrow{\text{add } R_2 \text{ to } R_3}$$

$$\cdots \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{mult } R_2 \text{ by } (-1/3)} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add } -2R_2 \text{ to } R_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows that the original linear system is equivalent to

$$\begin{array}{rcrcrcl} x & & & - & z & = & 0 \\ & & y & - & z & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

and, as discussed in 1.2, the solution set can be described parametrically as

$$x = z, \quad y = z, \quad z \text{ is free}$$

Described in this way, the solution set is difficult to visualize. We will discuss another way to describe the solution set, in terms of vectors, that will make it easier to visualize.

From a matrix-vector perspective, solving the linear system is finding a vector $v = (x, y, z)$ such that:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The above computations show that a vector $v = (x, y, z)$ in $\mathbb{R}^3$ is a solution to this equation if $x = z$ and $y = z$, where z is any real number, i.e.

$$v = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{for some } z \text{ in } \mathbb{R}$$

More succinctly, we can say that v is a solution if v is a scalar multiple of the vector $u = (1, 1, 1)$.

This perspective allows us to interpret the solution set geometrically: it is the line through the origin in $\mathbb{R}^3$ that passes through the point $(1, 1, 1)$.

Example 2 Consider the following (nonhomogeneous) linear system:

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 5 \\ 2x & + & y & - & 3z & = & 13 \\ -x & + & y & & & = & -8 \end{array}$$

To solve the linear system, we look at the corresponding augmented matrix and row reduce, as usual:

$$\begin{pmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{pmatrix} \xrightarrow{\text{add -2 } R_1 \text{ to } R_2} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ -1 & 1 & 0 & -8 \end{pmatrix} \xrightarrow{\text{add } R_1 \text{ to } R_3} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{pmatrix} \xrightarrow{\text{add } R_2 \text{ to } R_3}$$

$$\cdots \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{mult } R_2 \text{ by } (-1/3)} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add } -2R_2 \text{ to } R_1} \begin{pmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows that the original linear system is equivalent to

$$\begin{array}{rcrcrcr} x & & & - & z & = & 7 \\ & & y & - & z & = & -1 \\ & & & & 0 & = & 0 \end{array}$$

and, as discussed in 1.2, the solution set can be described parametrically as

$$x = z + 7, \quad y = z - 1, \quad z \text{ is free}$$

Described in this way, the solution set is difficult to visualize. We will discuss another way to describe the solution set, in terms of vectors, that will make it easier to visualize.

From a matrix-vector perspective, solving the linear system is finding a vector $v = (x, y, z)$ such that:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ -8 \end{pmatrix}$$

The above computations show that a vector $v = (x, y, z)$ in $\mathbb{R}^3$ is a solution to this equation if

$$v = \begin{pmatrix} z+3 \\ z+1 \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} + \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} \qquad \text{for some } z \text{ in } \mathbb{R}$$

More succinctly, we can say that v is a solution if

$$v = p + t\,u, \qquad \text{where } p = (7, -1, 0),\ u = (1, 1, 1), \text{ and } t \text{ is in } \mathbb{R}$$

This perspective allows us to interpret the solution set geometrically: it is the line through $(7, -1, 0)$ parallel to u. (Note that this is a translation by p of the solution set of the previous example!)

When we describe the solution set of a linear system explicitly in terms of vectors, as in Examples 1 and 2, this description is called *parametric vector form.*

The system in Example 1 is called *homogeneous* because the corresponding matrix-vector equation is of the form $Av = 0$.

Question: Can a homogeneous system be inconsistent? Why or why not?

> A homogeneous linear system cannot be inconsistent, because the the zero vector $v = 0$ is always a solution. Because of this, we call the zero vector the *trivial* solution of the homogeneous system, and we call any other solution a *nontrivial* solution.

Question: Does the homogeneous system in Example 1 have a nontrivial solution?

> Yes, because any (nonzero) scalar multiple of $u = (1, 1, 1)$ is a nonzero solution.

Fact: The solution set of a homogeneous linear system can always be expressed as the *span* of a suitable set of vectors. (Why?) This means that its geometric interpretation will be linear space (point, line, plane, or higher dimensional analogue) passing through the origin.

Theorem: The solution set of a *consistent* nonhomogeneous linear system is a translation of the solution set of the corresponding homogeneous linear system.