1 Overview

Main ideas:

1. linear transformation uniquely determined by its effect on $\{e_1, \ldots, e_n\}$; definition of standard matrix for linear transformation
2. geometric linear transformations of the plane (tables)
3. existence and uniqueness of solutions related to properties of linear transformation (onto, one-to-one)
4. Theorem: characterization of one-to-one linear transformations
5. Theorem: characterization of onto linear transformations

Examples in text:

1. determine Tv for arbitrary v if Te_1 and Te_2 are known (and $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation)
2. standard matrix for dilation (of plane) by a factor of 3
3. standard matrix for rotation (of plane) by arbitrary angle
4. determine whether a transformation given by a 3×4 matrix is one-to-one, onto
5. determine whether a transformation given by a formula is one-to-one, onto

2 Discussion and Worked Examples

2.1 The Matrix Associated to a Linear Transformation

Recall that a linear transformation of the plane is uniquely determined by its effect on $\hat{\imath}$ and $\hat{\jmath}$. This follows from linearity. More generally, a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined by its effect on the vectors $e_1, \ldots, e_n$, where

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, 0, \ldots, 0) \\ e_3 &= (0, 0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 0, 1) \end{aligned}$$

In other words, e_i is the vector in $\mathbb{R}^n$ whose only nonzero entry is its i-th entry, which is 1. Another way to think of this is that $e_1, \ldots, e_n$ are the columns of the $n \times n$ identity matrix, denoted $\mathbb{I}_n$.

Example Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and

$$Te_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad Te_2 = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$$

Find the image of v under T, where

$$v = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

The key is to notice that $v = 4e_1 + 7e_2$ and use linearity:

$$T(v) \;=\; T(4e_1 + 7e_2) \;=\; 4\,Te_1 \;+\; 7\,Te_2 \;=\; 4\begin{bmatrix}3\\2\\1\end{bmatrix} \;+\; 7\begin{bmatrix}-1\\0\\5\end{bmatrix} \;=\; \begin{bmatrix}12-7\\8+0\\4+35\end{bmatrix} \;=\; \begin{bmatrix}5\\8\\39\end{bmatrix}$$

Question Will this work for arbitrary v in $\mathbb{R}^2$? Why or why not?

Proof (that linearity implies that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by its effect on the vectors $e_1, \ldots, e_n$.)

> Let v be an arbitrary vector in $\mathbb{R}^n$. We will show that Tv can be expressed in terms of $Te_1, \ldots, Te_n$, thus proving the claim.
>
> First we note that we can write v uniquely as a linear combination of $e_1, \ldots, e_n$. One way to see this is to observe that the matrix-vector equation $\mathbb{I}_n x = v$ has a unique solution. (The solution exists for arbitrary v because $\mathbb{I}_n$ clearly has a pivot in each row. The solution is unique because every column is a pivot column, so there are no free variables.)
>
> So let $c_1, \ldots, c_n$ be weights such that $v \;=\; c_1e_1 \,+\, \ldots \,+\, c_ne_n$. Then observe that, by linearity,
>
> $$T(v) \;=\; T(c_1e_1 \,+\, \ldots \,+\, c_ne_n) \;=\; c_1\,T(e_1) \;+\; \ldots \;+\; c_n\,T(e_n)$$
>
> Thus the vector Tv in $\mathbb{R}^m$ can be written as a linear combination of the vectors $Te_1, \ldots, Te_n$.
>
> Since the weights $c_1, \ldots, c_n$ are uniquely determined, this equation determines Tv uniquely.

Observe that, in the proof above, the vector $(c_1, \ldots, c_n)$ is v written in coordinates, and it is also a solution to the matrix-vector equation $Ax = w$, where A is the matrix whose columns are $Te_1, \ldots, Te_n$ and w is the image Tv of v in $\mathbb{R}^m$. In other words, computing the matrix-vector product Av gives the coordinates of the vector Tv in $\mathbb{R}^m$. So A is just a representation of T in coordinates. It is called the *standard matrix* for T.

We omit the discussion of linear transformations of the plane, since we have already discussed several examples in depth.

2.2 One-to-one and Onto Linear Transformations

Recall that a function $f(x)$ is *one-to-one* if two distinct inputs x_1 and x_2 in the domain are always mapped forward to two distinct outputs y_1 and y_2 in the range, i.e. if $f(x_1) = f(x_2)$ then $x_1 = x_2$. (For functions whose domain and codomain is $\mathbb{R}$, we can use the horizontal line test to determine whether a function is one-to-one.) Another way to say this is that, for any y in the range, the equation $f(x) = y$ is satisfied by a unique x in the domain.

A function $f(x)$ is *onto* if, for every y in the codomain there exists an x in the domain such that $f(x) = y$. In other words the range is the whole codomain.

In the context of linear transformations, these properties are relevant to existence and uniqueness questions. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and A its standard matrix. If T is one-to-one, then $Ax = b$ has a unique solution, if it is consistent. If T is onto, then for all b in $\mathbb{R}^m$, $Ax = b$ has a solution.

In particular, Theorem 4 (in Section 1.4) is all about onto linear transformations. A linear transformation is onto if and only if its standard matrix has a pivot position in every row.

It turns out that a linear transformation T is one-to-one if and only if the homogeneous equation $Ax = 0$, where A is the standard matrix for T, has only the trivial solution. Thus, T is one-to-one if and only if the columns of A are linearly independent.

Proof (of the characterization of one-to-one linear transformations above)

First note that zero is in the range of every linear transformation, because the image of zero is always zero.

Now, let be T a one-to-one linear transformation, and let A be its standard matrix. Then, for each w in the range of T, the equation $T(v) = w$ has a unique solution v in the domain. In particular, $Tv = 0$ has a unique solution v. But, of course, $T(0) = 0$; thus zero is the only solution to $Tv = 0$. Since $T(v) = Av$, this means that the matrix-vector equation $Av = 0$ has only the trivial solution.

We also need to show the converse, namely that if $Av = 0$ has only the trivial solution, T is one-to-one. Let w be an arbitrary vector in the range of T. We need to show that the equation $Tv = w$ is satisfied by a unique v in the domain. Of course, if $w = 0$, we are done, so it suffices to consider $w \neq 0$. Recall that the solution set to the non-homogeneous equation $Av = b$ is a translation of the solution set of the homogeneous equation $Av = 0$ (as long as the non-homogeneous equation is consistent.) Since the solution to the homogeneous equation is unique, the solution to the non-homogeneous system is also unique.

In summary:

T is 1-1 $\Rightarrow$ $Ax = 0$ has only the trivial sol'n

$Ax = 0$ has only the trivial sol'n $\Rightarrow$ if $Ax = b$ has a sol'n, it's unique $\Rightarrow$ T is 1-1

Example Consider the linear transformation T whose standard matrix is

$$A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix} \longrightarrow \dots \longrightarrow \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(row reduction)}$$

Is T one-to-one? onto?

Since A has a pivot position in each row, T is onto.

Since $Av = 0$ has nontrivial solutions, namely all vectors of the form

$$v = t \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad \text{for } t \text{ in } \mathbb{R}$$

T is not one-to-one.