1 Overview

Main ideas:

1. definition of an abstract vector space; simple facts that hold in an arbitrary vsp
2. definition of a subspace; every ssp is a vsp in its own right
3. definitions of linear combinations, span of a set, subspace spanned by (or generated by) a set, spanning (or generating) set; the span of a set of vectors is s ssp

Examples in text:

1. $\mathbb{R}^n$, $n \geq 1$ is a vsp
2. arrows in 3D space (geom model for $\mathbb{R}^3$) is a vsp
3. the set, $\mathbb{S}$, of signals is a vsp
4. the set, $\mathbb{P}_n$, of polynomials of degree $\leq n$, along with the zero polynomial $P(x) = 0$, whose degree is undefined, is a vsp
5. the set of all real-valued fcns on a certain domain is a vsp
6. zero ssp
7. the vsp $\mathbb{P}_n$ is a ssp of the vsp, $\mathbb{P}$, consisting of all polynomials with real coefficients
8. while $\mathbb{R}^2$ is not technically a ssp of $\mathbb{R}^3$, since it is not a subset even, there are "copies" of $\mathbb{R}^2$ that are ssp of $\mathbb{R}^3$
9. a plane in $\mathbb{R}^3$, not passing through the origin is *not* a ssp of $\mathbb{R}^3$, and a line in $\mathbb{R}^2$, not passing through the origin is *not* a ssp of $\mathbb{R}^2$
10. show that the span of a set of two vectors is a ssp
11. show that a certain set is a ssp by demonstrating that the set is the span of a set of vectors
12. determine whether a vector is in the ssp spanned by a given set

2 Discussion and Worked Examples

2.1 Definition of an Abstract Vector Space

Let V be a (nonempty) set. Let $+$ (or **addition**) be an operation on V, i.e.

- For elements v, w in V, the sum $v + w$ is well defined and is another element of V.

satisfying the following properties:

- There exists in V an **additive identity** element, i.e. an element v_0 such that

$$v_0 + v \;=\; v + v_0 \;=\; v \qquad \text{for all } v \text{ in } V$$

- For each v in V, there is an **additive inverse** in V, i.e. an element $\tilde{v}$, such that

$$v \,+\, \tilde{v} \;=\; \tilde{v} \,+\, v \;=\; 0$$

- Addition is **associative**, i.e. for all u, v, w in V,

$$(u + v) + w \;=\; u + (v + w)$$

- Addition is **commutative**, i.e. for each v, w in V,

$$v + w \; = \; w + v$$

Let the real numbers act on V by **scalar multiplication**, i.e.

- For any real number c, cv is well defined and is an element of V.
- Scalar multiplication is **associative**: i.e. for any real numbers c and d and any element v of V,

$$(cd)\,v \; = \; c\,(dv)$$

- Scalar multiplication is **distributive**: for any real numbers c and d and any elements v, w of V,

$$c(v + w) \; = \; cv + cw \qquad \text{and} \qquad (c + d)v = cv + dv$$

- Scalar multiplication by 1 is the **identity** map, i.e. for every v in V,

$$1v = v$$

Then we say that V is a (real) **vector space**, and we call the elements of V vectors.

2.2 Examples of Vector Spaces

Certainly $\mathbb{R}^n$, with addition and scalar multiplication as defined in Chapter 1, is a vector space.

Other examples include:

1. $\mathbb{S}$, the set of all doubly infinite sequences, with addition and scalar multiplication defined element-wise
2. $\mathbb{P}_n$, the set of all polynomials with real coefficients of degree at most n, with addition and scalar multiplication being the usual addition and constant multiple of functions
3. the set of all real-valued functions on a given domain.

We discuss this last example in some detail. Let V be the set of all functions on the domain $[0, 1]$. In order for V to be a vector space, we need to be able to add functions and scale functions by real numbers, and we need these operations to satisfy the properties outlined above.

We define addition of functions pointwise: given two functions f and g on $[0, 1]$, the function $f + g$ is defined by:

$$(f + g)(x) \; = \; f(x) \; + \; g(x) \qquad \text{for all } x \text{ in } [0, 1]$$

At first glance, this may appear to be a meaningless statement, due to the fact that, in common notation, a function f is not distinguished from its values (outputs) $f(x)$, but here we do make this distinction. Thus, on the left side of the equation, we are adding *functions*, while on the right side we are adding *numbers*.

Question Can you think of a function that is an additive identity?

This would need to be a function f_0 that has the following property: for any function f on $[0, 1]$

$$f_0 + f \; = \; f + f_0 \; = f$$

i.e.

$$f_0(x) \; + \; f(x) \; = \; f(x) \; + \; f_0(x) \; = \; f(x) \qquad \text{for all } x \text{ in } [0, 1]$$

This could only be the zero map: $f_0 : x \mapsto 0$.

Question Given a function f on $[0,1]$, what function would be an additive inverse?

We need a function g such that
$$f+g \;=\; g+f \;=\; f_0$$
i.e.
$$f(x) + g(x) \;=\; g(x) + f(x) \;=\; f_0(x) \qquad \text{for all } x \text{ in } [0,1]$$
Since $f_0(x) = 0$ for all x, this means we need $f(x)+g(x) = g(x)+f(x) = 0$ for all x, i.e. we need $g(x) = -f(x)$ for all x. So this is how we define the additive inverse for f: it is the function $x \mapsto -f(x)$.

Associativity and commutativity of addition of function follows from associativity and commutativity of real numbers, respectively. For example, we prove $(f+g)+h = f+(g+h)$, as follows. For any x in $[0,1]$, by the definition of addition of functions,

$$\big((f+g)+h\big)(x) \;=\; (f+g)(x) + h(x) \;=\; \big(f(x)+g(x)\big) + h(x)$$

and by associativity of addition of real numbers,

$$\big(f(x)+g(x)\big) + h(x) \;=\; f(x) + \big(g(x)+h(x)\big)$$

and, again, by the definition of addition of functions,

$$f(x) + \big(g(x)+h(x)\big) \;=\; f(x) + (g+h)(x) \;=\; \big(f + (g+h)\big)(x)$$

and thus
$$\big((f+g)+h\big)(x) \;=\; \big(f + (g+h)\big)(x) \qquad \text{for all } x \text{ in } [0,1]$$
i.e. $(f+g)+h = f+(g+h)$.

Similarly, we define scalar multiplication on functions pointwise: the scalar multiple of a function f by a real number c is defined by:
$$(c\,f)(x) \;=\; c\,f(x) \qquad \text{for every } x \text{ in } [0,1]$$
Again, on the left side of the equation we are scaling a function by a real number, but on the right side of the equation we are simply multiplying two real numbers. The desired properties of scalar multiplication follow from the corresponding properties of real numbers.

Notice that the fact that we chose $[0,1]$ as our domain was not important. For any domain D in $\mathbb{R}$, the set of functions on D is a vector space.

2.3 Simple Consequences of the Definition

The following results follow directly from the definition of an abstract vector space.

1. The additive identity is unique, i.e if there are two vectors in V that function as additive identities, then they are equal.
2. Additive inverses are unique, i.e. given a vector v in V, if two vectors are additive inverses for v, then they are equal.
3. Scaling a vector by zero results in the additive identity vector.
4. Scaling the additive identity vector by any number results in the additive identity vector.
5. Scaling a vector v by -1 results in the additive inverse of v.

We will prove (1) and (4). The others are proven in homework exercises.

To prove (1), suppose that v_0 and w_0 are both additive identities. Then $v_0 + w_0 = v_0$, since w_0 is an additive identity. On the other hand, $v_0 + w_0 = w_0$, since v_0 is an additive identity. We have shown

$$v_0 \;=\; v_0 + w_0 \;=\; w_0$$

Thus $v_0 = w_0$, proving that the additive identity is unique.

Note Since there is only one vector in a given vector space that functions as an additive identity, we may give it a name. Not surprisingly, we call it the **zero vector**, and, depending on the context, we might even denote it as 0, as in $\mathbb{R}^n$.

Now we prove (4), assuming (3). Let v_0 be the zero vector and c be any scalar. By (3), $0v_0 = v_0$, so

$$cv_0 \;=\; c(0v_0) \;=\; (c \cdot 0)v_0 \;=\; 0v_0 \;=\; v_0$$

using the fact that $c(dv) = (cd)v$ for any two scalars c and d and any vector v, as well as using (3) again. Thus scaling the zero vector always results in the zero vector.

Note Consider a vector v in an abstract vector space. By (2), there is only one vector that functions as an additive inverse for v. By (5), we can find the additive inverse by simply scaling v by (-1). Because of this, we denote the additive inverse for v by $-v$.

2.4 Subspaces: Definition and Examples

Definition A subset W of a vector space V is a *subspace* of V if

1. The zero vector lies in W.
2. The set W is closed under vector addition, i.e. the sum of any two vectors in W lies in W.
3. The set W is closed under scalar multiplication, i.e. any scalar multiple of a vector in W lies in W.

Notes (1) If W is a subspace of a vector space V, then W is a vector space in its own right. (2) The subset of a vector space consisting of solely the zero vector is always a subspace. (3) Every vector space is a subspace of itself.

(Examples and non-examples activity.)

2.5 The Span of a Set of Vectors

A linear combination of vectors $v_1, \ldots, v_p$ in a vector space V is a sum of scalar multiples of those vectors: $c_1v_1 + \ldots + c_pv_p$. The span of a set $\{v_1, \ldots, v_p\}$ is the set of all linear combinations of $v_1, \ldots, v_p$:

$$\text{Span}\{v_1, \ldots, v_p\} \;\stackrel{\text{DEF}}{=}\; \{w \text{ in } V \text{ such that } w \;=\; c_1v_1 + \ldots + c_pv_p \text{ for some } c_1, \ldots, c_p \text{ in } \mathbb{R}\}$$

The span of a set of vectors is a subspace; in fact it is the smallest subspace containing the specified vectors. If $W = \text{Span}\{v_1, \ldots, v_p\}$, we call W the subspace spanned by $\{v_1, \ldots, v_p\}$, and we call $\{v_1, \ldots, v_p\}$ the spanning set of W.

This can be a useful way to prove that a given set W is a subspace, by identifying a set of vectors of which W is the span.