Definition. Let D be a domain, and let $X = \{(a, b) : a, b \in D, b \neq 0\}$. Define a relation $\equiv$ on X by: $(a, b) \equiv (c, d)$ if and only if $ad = bc$. This is an equivalence relation (Lemma 5.4). For $(a, b) \in X$, let $[a, b]$ denote the equivalence class of (a, b), i.e.

$$[a, b] \;=\; \{(c, d) \in X : (a, b) \equiv (c, d)\} \;=\; \{(c, d) : c, d \in D, d \neq 0, ad = bc\}$$

Define the fraction field of D, denoted $\text{Frac}(D)$ to be the set of equivalence classes of elements of X:

$$\text{Frac}(D) \;=\; \{[a, b] : (a, b) \in X\} \;=\; \{[a, b] : a, b \in D, b \neq 0\}$$

We define addition and multiplication on $\text{Frac}(D)$ by $[a, b] + [c, d] = [ad + bc, bd]$ and $[a, b] \cdot [c, d] = [ac, bd]$. These operations are well-defined (Theorem 5.5(i)).

With the addition and multiplication as defined above, the fraction field of a domain is a field, with the additive identity being $[0, 1]$, the multiplicative identity being $[1, 1]$, the additive inverse of an element $[a, b]$ being $[-a, b]$, and the multiplicative inverse of an element $[a, b]$ being $[b, a]$ (Theorem 5.5(i)).

Identifying $a \in D$ with the equivalence class $[a, 1] \in \text{Frac}(D)$, we consider D to be a subring of $\text{Frac}(D)$. Since D is not literally a subset of $\text{Frac}(D)$, it cannot literally be a subring of $\text{Frac}(D)$. To be precise we should say that D is *isomorphic* to the subset $D' = \{[a, 1] : a \in D\}$ of $\text{Frac}(D)$, which is, literally, a subring of $\text{Frac}(D)$ (Theorem 5.5(ii)).

Example 1. We can construct $\mathbb{Q}$ from $\mathbb{Z}$ in this way, since $\mathbb{Z}$ is a domain. If we define $\mathbb{Q} = \text{Frac}(\mathbb{Z})$, then $\mathbb{Z}$ is not literally a subset of $\mathbb{Q}$, since $\mathbb{Q}$ consists of equivalence classes of certain pairs of integers. In practice, however, we usually choose not to distinguish between $\mathbb{Z}$ and its isomorphic copy $\{[a, 1] : a \in \mathbb{Z}\}$ inside $\mathbb{Q}$.

Example 2. What is $\text{Frac}(\mathbb{Q})$? According to the definition,

$$\text{Frac}(\mathbb{Q}) \;=\; \{[r, s] : r, s, \in \mathbb{Q}, s \neq 0\} \qquad \text{where} \qquad [r, s] = \{(r', s') : r', s' \in \mathbb{Q}, s' \neq 0, \text{ and } rs' = sr'\}$$

Let (r, s) with $r, s \in \mathbb{Q}$ and $s \neq 0$. We claim that $(r, s) \equiv (t, 1)$, for some $t \in \mathbb{Q}$, namely $t = s^{-1}r$. To prove this, we show $r \cdot 1 = s \cdot t$, as follows:

$$r \cdot 1 \;=\; 1 \cdot r \;=\; (s \cdot s^{-1})r \;=\; s \cdot (s^{-1}r) \;=\; s \cdot t.$$

Thus, for any $[r, s] \in \text{Frac}(\mathbb{Q})$, there is an element $t \in \mathbb{Q}$ such that $[r, s] = [t, 1]$. This shows that *every* element of $\text{Frac}(\mathbb{Q})$ is contained in the set $\{[t, 1] : t \in \mathbb{Q}\}$, which is the isomorphic copy of $\mathbb{Q}$ inside $\text{Frac}(\mathbb{Q})$. Of course, $\{[t, 1] : t \in \mathbb{Q}\} \subset \text{Frac}(\mathbb{Q})$, by definition, so we have shown that $\text{Frac}(\mathbb{Q}) \;=\; \{[t, 1] : t \in \mathbb{Q}\}$, i.e. the fraction field of $\mathbb{Q}$ is (isomorphic to) $\mathbb{Q}$ itself.

Note. The above argument works with $\mathbb{Q}$ replaced by any field; thus the fraction field of a field k is (isomorphic to) k itself. This should not be surprizing, since the construction of $\text{Frac}(D)$ from a domain D amounts to constructing reciprocals (well, multiplicative inverses) for every element of D.

Example 3. What is $\text{Frac}(\mathbb{Z}[i])$? According to the definition,

$$\text{Frac}(\mathbb{Z}[i]) \;=\; \{[z, w] : z, w \in \mathbb{Z}[i], w \neq 0\} \qquad \text{where} \qquad [z, w] \;=\; \{[z', w'] : z', w' \in \mathbb{Z}[i], w' \neq 0, \text{ and } zw' = wz'\}$$

We might guess that $\text{Frac}(\mathbb{Z}[i]) = \mathbb{Q}(i)$, where $\mathbb{Q}(i) = \{r + si : r, s \in \mathbb{Q}\}$. We will show that this is, in fact, true, as long as we're willing to identify elements of $\mathbb{Q}(i)$ with equivalence classes of pairs of Gaussian integers (just as we identify rational numbers with equivalence classes of pairs of integers.)

Since multiplication in $\mathbb{Z}[i]$ is the multiplication defined on $\mathbb{C}$, by Theorem 5.5(iii), we can identify $\text{Frac}(\mathbb{Z}[i])$ with the following subset of $\mathbb{C}$:

$$\left\{\frac{z}{w} : z, w \in \mathbb{Z}[i], w \neq 0\right\}$$

Take any $z, w \in \mathbb{Z}[i]$. Then $z = a + bi$ and $w = c + di$ for some $a, b, c, d \in \mathbb{Z}$, and

$$\frac{z}{w} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right) i \in \mathbb{Q}(i)$$

On the other hand, if $r + si \in \mathbb{Q}(i)$, then $r = p/q$ for some $p, q \in \mathbb{Z}$ and $s = p'/q'$ for some $p', q' \in \mathbb{Z}$, and

$$r + si = \left(\frac{p}{q}\right) + \left(\frac{p'}{q'}\right) i = \left(\frac{pq'}{qq'}\right) + \left(\frac{p'q}{qq'}\right) i = \frac{(pq)' + (p'q)i}{qq'}$$

where both the numerator $(pq)' + (p'q)i$ and denominator qq' are in $\mathbb{Z}[i]$.

Thus, we have the following equality of subsets of $\mathbb{C}$:

$$\left\{\frac{z}{w} : z, w \in \mathbb{Z}[i], w \neq 0\right\} = \{r + si : r, s \in \mathbb{Q}\} = \mathbb{Q}(i)$$

and we can identify $\text{Frac}(\mathbb{Z}[i])$ with $\mathbb{Q}(i)$.

Exercise 5.6(ii) Is $\text{Frac}(\mathbb{Z}[i]) = \{[r + si, 1] : r, s \in \mathbb{Q}\}$? Yes and no.

This is formally incorrect, because for $[r + si, 1]$ to be in $\text{Frac}(\mathbb{Z}[i])$, we would need $r + si \in \mathbb{Z}[i]$, which is not necessarily the case.

However, as discussed in Example 3, above, $\text{Frac}(\mathbb{Z}[i])$ can be identified with $\mathbb{Q}(i)$, and, by the note after Example 1, above, since $\mathbb{Q}(i)$ is a field, $\text{Frac}(\mathbb{Q}(i)) = \{[r + si, 1] : r, s \in \mathbb{Q}\}$, which can also be identified with $\mathbb{Q}(i)$. Thus, although formally $\text{Frac}(\mathbb{Z}[i])$ is not equal to $\{[r + si, 1] : r, s \in \mathbb{Q}\}$, both sets can be identified with $\mathbb{Q}(i)$, so the two sets can be identified with each other.