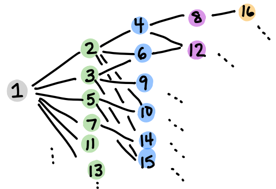
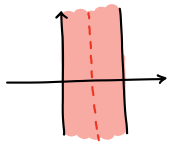
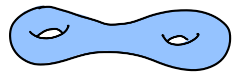
Heat Kernels, Spectral Theory, & Zeta Functions

① Number Theory & Zeta Functions





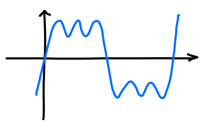
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

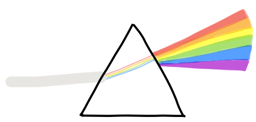


$$Z(s) = \prod_{\{p\}} \prod_{k=0}^{\infty} \left(1 - N(p)^{-s-k}\right)$$

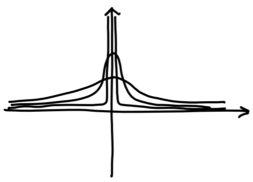
② Mathematical Physics & Spectral Theory

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$





③ Heat Kernels & Zeta Functions

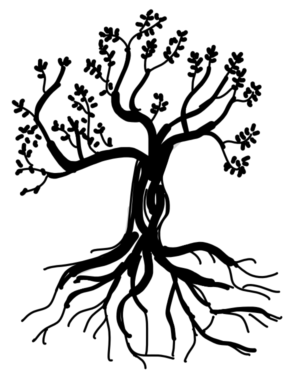


$(\Delta - \partial_t)u = \delta$

$$\xi(s) = \int_0^{\infty} y^{s} \, \frac{\theta(y)-1}{2} \, \frac{dy}{y}$$

$$\frac{Z'(s)}{Z(s)} = (2s-1) \int_0^{\infty} \vartheta(t)\, e^{-s(s-1)}\, dt$$

Number Theory



Natural Numbers

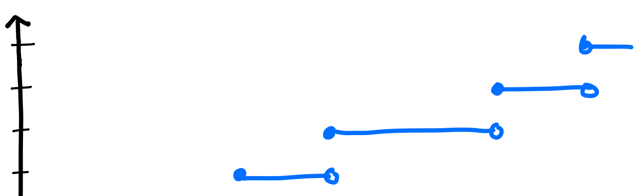
1, 2, 3, 4, 5, 6, ...

Prime Numbers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

- How many? (Euclid)
- How frequent? Distribution?

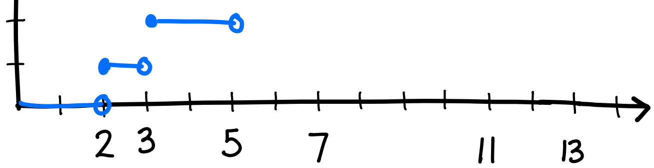
$\pi(X)$ = # primes $\leq X$ ("staircase")



Legendre (1798)

$\pi(X) \sim \frac{X}{\quad}$

$\ln X$



Pf : Hadamard & de la Valée Poussin (1896)

"Prime Number Theorem"

Riemann Zeta Function

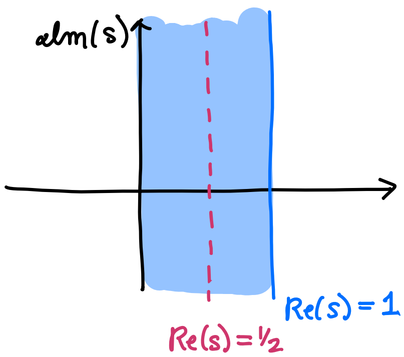
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - 1/p^s} \qquad (\mathrm{Re}(s) > 1)$$

$s = 2$, $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$

$s = 1$ $\quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

- Euler (FTA, $\zeta(1) = \infty \Rightarrow \infty$'ly many primes)
- Riemann (1858, $s \in \mathbb{C}$, explicit fmla, RH)

1, 2, 3, 4, 5, 6,
7, 8, 9, 10, 11, ...

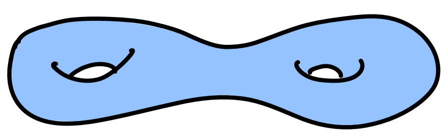


$\zeta(\rho) = 0$
$\Rightarrow \mathrm{Re}(\rho) = \frac{1}{2}$?

zeta as integral involving
"theta", an "automorphic form"

Selberg's Zeta Function

$$Z(s) = \prod_{\{p\}} \prod_{k=0}^{\infty} \left(1 - N(p)^{-s-k}\right) \qquad (\mathrm{Re}(s) > 1)$$



Selberg Trace Formula

lengths of closed primitive geodesics on $M = \Gamma \backslash \mathbb{H}$ $\longleftrightarrow$ eigenvals of Δ on M

* resembles RVM Explicit Fmla *

Mathematical Physics

Partial Differential Equations

$(\Delta - \lambda^2)\, u = \delta$

Schrödinger Equation

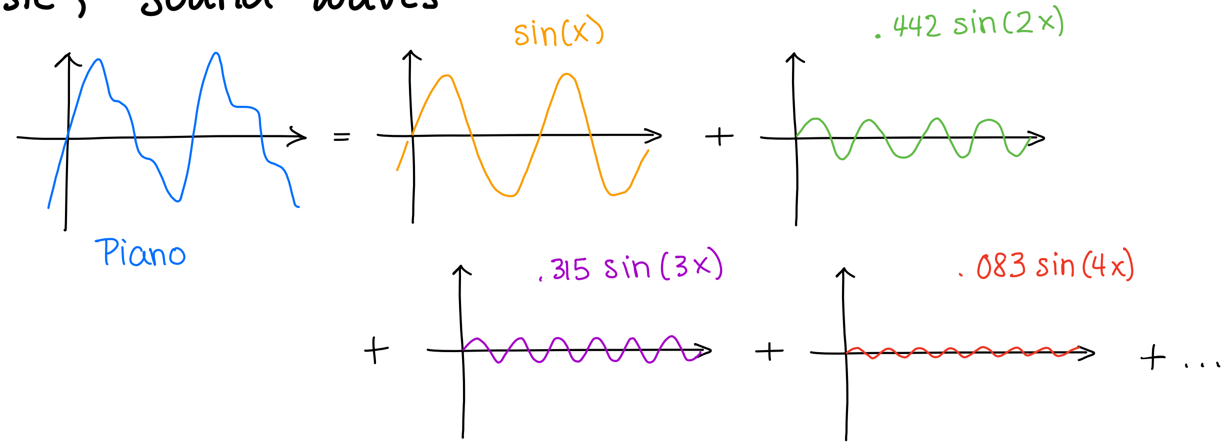
Heat Eqtn $(\Delta - \partial_t)\, u = \delta$ source

spacial deriv.

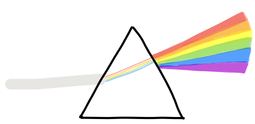
deriv. w/ resp to time

Fourier Transform

Music, sound waves



Light, color



Mathematically:

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

Fourier Series:

$$\sum_{\xi \in \mathbb{Z}} \hat{f}(\xi)\, e^{2\pi i x \xi}$$

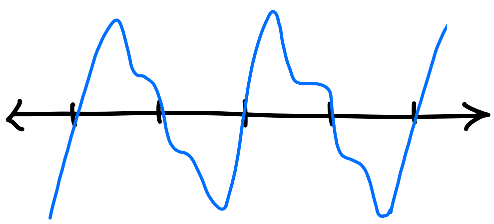
If f not periodic ... Fourier Inversion :

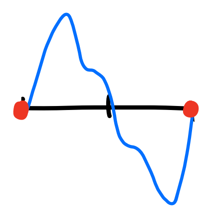
$$\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

PDEs : Differentiation $\longrightarrow$ Multiplication

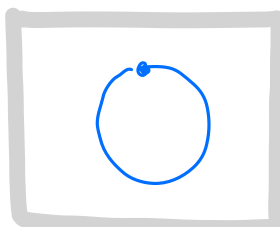
Solve PDEs by Division

Two Perspectives on Periodicity & Other Symmetries

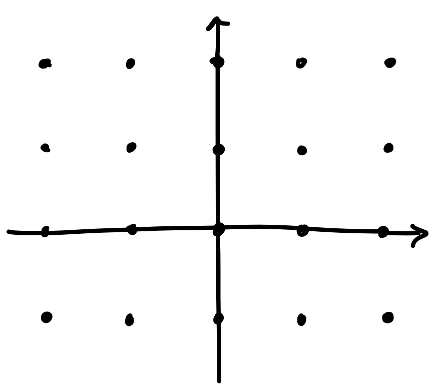




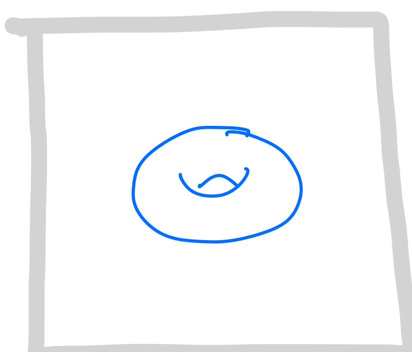
$\mathbb{R}$



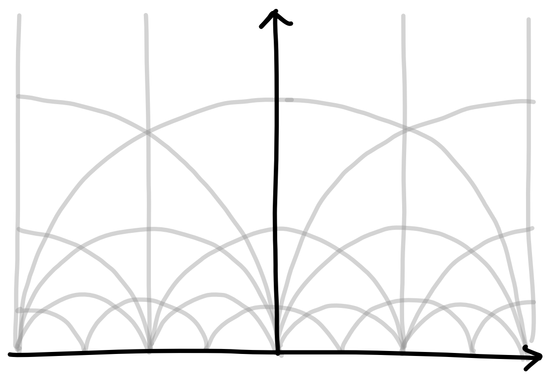
$\mathbb{R}/\mathbb{Z}$



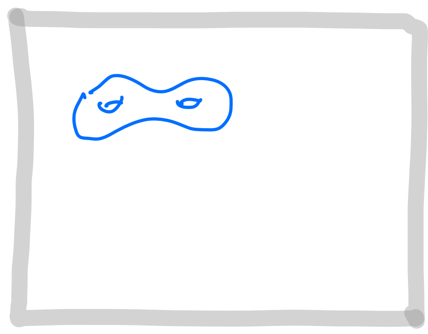
$\mathbb{R}^2$



$\mathbb{R}^2/\mathbb{Z}^2$



$\mathfrak{h} = \{x+iy \mid y>0\}$

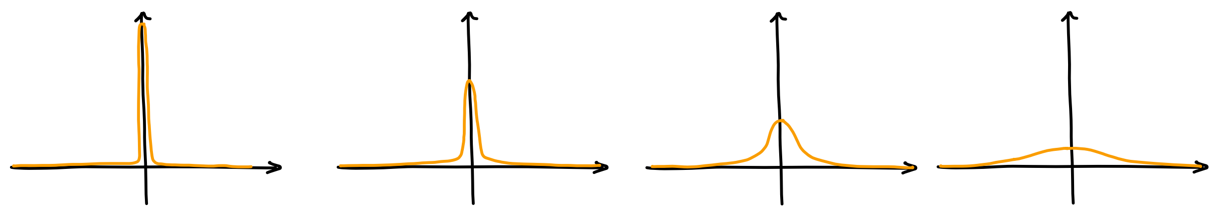


$\Gamma \backslash \mathfrak{h}$

Heat Kernels & Zeta Functions

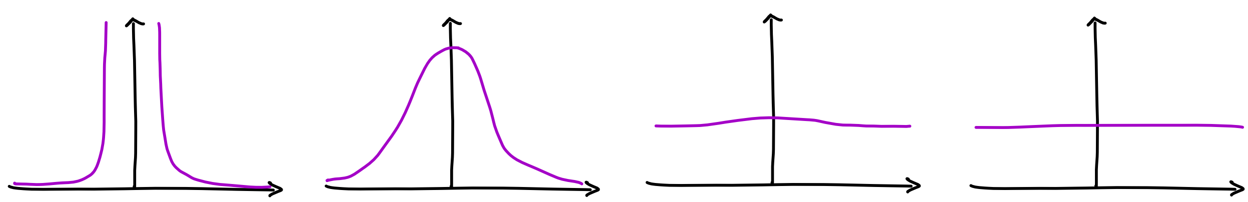
Heat Kernel: Solution to $(\Delta - \partial_t)u = \delta$

On $\mathbb{R}$:



$t = .001$ $\quad t = .01$ $\quad t = .1$ $\quad t = 1$

On $\mathbb{R}/\mathbb{Z}$ (or $[-\frac{1}{2}, \frac{1}{2}]$)



$t = .001$ $\quad t = .01$ $\quad t = .1$ $\quad t = 1$

Using F.I. & "Periodicization"

$$u_t^{\mathbb{R}}(x) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}} \longrightarrow u_t^{\mathbb{R}/\mathbb{Z}} = \sum_{n\in\mathbb{Z}} \frac{e^{-(x+n)^2/4t}}{2\sqrt{\pi t}}$$

Using F.S.

$u_t^{\mathbb{R}/\mathbb{Z}}$

$$u_t^{\mathbb{R}/\mathbb{Z}}(x) = \underbrace{\sum_{\xi\in\mathbb{Z}} e^{-4\pi^2\xi^2 t}\, e^{2\pi i \xi x}}_{\vartheta(x,\,4\pi t i)}$$

Putting them together $\Rightarrow$ Theta Inversion Formula

$$\theta(z) = \frac{1}{\sqrt{-iz}}\,\theta\left(\frac{1}{z}\right)$$

Meromorphic Continuation & Functional Equation for ξ

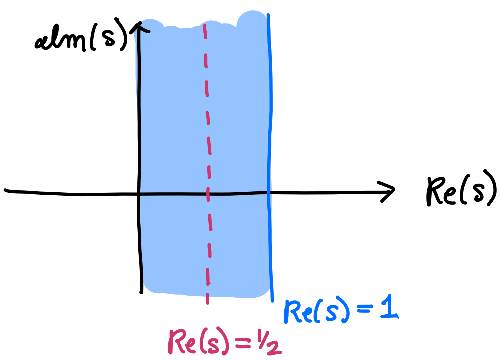
$$\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) \qquad \text{"completed" zeta}$$

Integral Representation:

$$\xi(s) = \int_0^\infty \left(\frac{\theta(iy)-1}{2}\right) y^{s/2}\,\frac{dy}{y}$$

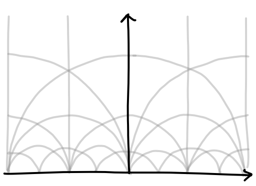
$\vdots$

$$\xi(s) = \xi(1-s)$$



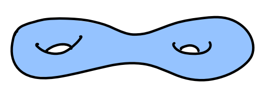
McKean: Selberg Zeta

Heat kernel on $\Gamma\backslash\mathfrak{h}$



$\mathfrak{h} = \{x+iy \mid y>0\}$

on $\mathfrak{h}$ $\xrightarrow{\text{"periodicize"}}$ on $\Gamma\backslash\mathfrak{h}$



$\Rightarrow$ "Theta Relation"

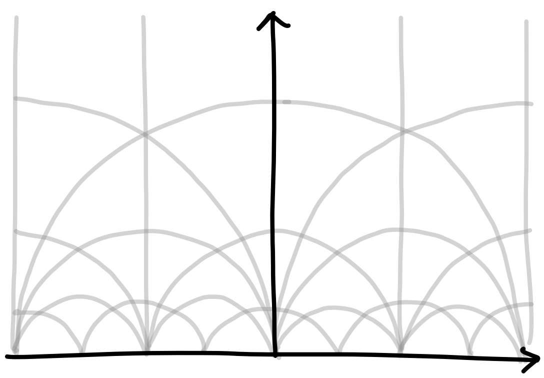
$\Big\downarrow$ Integral Transform

$$\frac{Z'(s)}{Z(s)} = (2s-1) \int_0^\infty \vartheta(t)\, e^{-s(s-1)t}\, dt$$

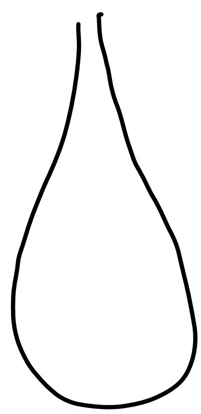
$$\vdots$$

$$Z(1-s) = Z(s) \cdot e^{2(g-1)\int_0^{s-\frac{1}{2}} \pi x \tan(\pi x)\, dx}$$

Using Spectral Theory of Automorphic Forms



$\leadsto$



$\Gamma \backslash \mathfrak{h}$

$\mathfrak{h} = \{x+iy \mid y>0\}$

& higher dimensional analogues :

$X = G/K \quad \leadsto \quad \Gamma \backslash X$

"symmetric space"

(arithmetic) quotient

harmonic analysis on symm. spaces

automorphic spectral theory

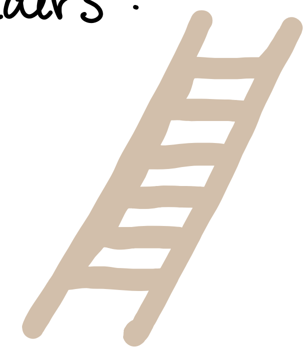
Solve PDE $\quad \cdot\ (\Delta - \lambda) u = \delta \quad$ in 2 ways.

$\qquad\qquad\quad \cdot\ (\Delta - \partial_t) u = \delta$

$\Rightarrow$ Spectral Identities

(ex: generalized "theta inversion")

Ladders :



$$\begin{array}{ccc} \vdots & & \vdots \\ G_3/K_3 & & Z_3 \\ | & & | \\ G_2/K_2 & & Z_2 \\ | & & | \\ G_1/K_1 & & Z_1 & \quad \zeta \end{array}$$

Legitimize use of techiques designed for $\mathbb{R}$ & $\mathbb{R}^n$?

- Functional Analysis
- Automorphic Sobolev Theory
- Semigroup Theory ...