The Automorphic Heat Kernel & Ladders of Zeta Functions

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1D Euclidean Heat Kernel

Heat kernel $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ satisfies:

$$(\partial_t - \Delta)\, u = 0, \qquad \lim_{t \to 0^+} u(x,t) = \delta.$$

Apply Fourier transform $\mathcal{F}$:

$$(\partial_t + 4\pi^2 \xi^2)\, \mathcal{F}u = 0, \qquad \lim_{t \to 0^+} (\mathcal{F}u)(\xi, t) = \mathcal{F}\delta = 1.$$

Considering ξ as fixed, $\mathcal{F}u(\xi, t)$ satisfies familiar IVP:

$$\frac{dy}{dt} = -4\pi^2 \xi^2\, y, \quad y(0) = 1 \quad \Rightarrow \quad y(t) = e^{-4\pi^2 \xi^2 t}$$

Fourier inversion: $u(x,t) = (4\pi t)^{-1/2}\, e^{-x^2/4t}$.

Theta Inversion and Riemann Zeta

Construct heat kernel $h_t^{\mathbb{R}/\mathbb{Z}}(x)$ on $\mathbb{R}/\mathbb{Z}$ by periodicization and by Fourier series:

$$\sum_{n\in\mathbb{Z}} \frac{e^{-(x+n)^2/4t}}{2\sqrt{\pi t}} = \sum_{\xi\in\mathbb{Z}} e^{-4\pi^2\xi^2 t}\, e^{2\pi i x\xi} .$$

Theta Inversion ($x = 0$ and $z = 4\pi i t$):

$$\theta(z) = \frac{1}{\sqrt{-iz}}\, \theta\left(\frac{1}{z}\right) .$$

Apply Mellin transform to:

$$\frac{\theta(iy)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 y} .$$

Mero. continuation and functional equation for Riemann zeta.

Selberg Zeta (McKean, 1972)

Compact Riemannian surface $M = \Gamma\backslash\mathfrak{H}$ (genus > 1).

Two-variable heat kernel on $\mathfrak{H}$:

$$K_t(x,y) \;=\; \frac{e^{-t/4}\sqrt{2}}{(4\pi t)^{3/2}} \int_a^\infty \frac{b\, e^{-b^2/4t}}{\sqrt{\cosh b - \cosh a}}\, db \;,$$

where $a = d(x,y)$ is the distance between x and y in $\mathfrak{H}$.

Heat kernel on M:

$$K_t^M(x,y) \;=\; \sum_{\gamma\in\Gamma} K_t(x,\gamma y)\;.$$

Selberg Trace Formula

Expand trace of K_t^M in two different ways:

$$\sum_{n=0}^{\infty} e^{\lambda_n t} = \text{area}(M) \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{b e^{-b^2/4t}}{\sinh \frac{1}{2} b} \, db$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\{p\}} \frac{\ell(p)}{\sinh \frac{1}{2} \ell(p^n)} \frac{e^{-t/4}}{(4\pi t)^{1/2}} e^{-|\ell(p^n)|^2/4t} .$$

RHS: "trivial term" $+ \vartheta(t)$.

Selberg Zeta

Integral Transform:

$$f \;\mapsto\; (2s-1)\int_0^\infty f(t)\, e^{-s(s-1)t}\, dt\,.$$

Apply to theta function, spectral side, and trivial term:

$$\frac{Z'(s)}{Z(s)} \;=\; \sum_{n=0}^{\infty}\left(\frac{1}{s-s_n} + \frac{1}{s-(1-s_n)}\right)$$

$$-\;2(g-1)\;\sum_{k=0}^{\infty}\frac{2s-1}{s+k}\,.$$

Mero. continuation and functional equation for Selberg zeta.

Jorgenson-Lang Vision

To create new zeta functions:

1. Start with the heat kernel on a symmetric space G/K.
2. Periodicize it with respect to a discrete subgroup Γ.
3. Determine the eigenfunction expansion on $\Gamma\backslash G/K$.
4. Regularize the expansion and integrate over $\Gamma\backslash G$.
5. Apply Gauss transform: $f \mapsto 2s\int_0^\infty f(t)\, e^{-s^2 t}\, dt$.

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   Effect? On geometric side: Apply to heat kernel

   - on $\mathbb{R}$: exponential; wind up: Dirichlet series.
   - on $\mathbb{R}^n$: K-Bessel function (elem. if n odd)
   - on G/K, G cx: poly. times K-Bessel

   On spectral side:

   - Apply to terms of the form $a_\xi e^{-\lambda_\xi t}$ with $\lambda_\xi = \xi^2$.
   - Rational functions with simple poles at $\pm i\xi$.
   - Sum is log derivative of function with zeros at $\pm i\xi$ of mult. a_ξ?

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Jorgenson and Lang

- Compact quotient of $\mathbb{H}^3$ (1994, 1996)
    - Shintani and Millson (Millson, 1978)
- Eventually main theme of joint research (1993-2012).

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Such zeta functions:

- Of interest in themselves.
- Perhaps shed light on other zeta functions of interest.
  - Consider ladders of symmetric spaces.
  - Ex: natural embedding $SL_n(\mathbb{C})/SU(n) \hookrightarrow SL_{n+1}(\mathbb{C})/SU(n+1)$.
  - Corresponding ladders of zeta functions.

Challenges

1. Heat kernel on G/K is not elementary; asymptotics nontrivial.
   - Wind up? (Convergence?)
2. Eigenfunction expansion on $\Gamma\backslash G/K$? (Convergence?)
3. Regularization? Trace? Alternatives?
   - Pre-trace formula?
   - Evaluation?
4. Integral transform (Convergence? Modifications?)
   - What do we get?

Challenge 1

Wind-up heat kernel on G/K?

- Gangolli 1968: Bi-K-invariant heat kernel on G, conn. ss. Lie, finite center.
  - Integral representation:

$$h_t(a) \ = \ \int_{\mathcal{W}\backslash\mathfrak{a}^*} e^{-t(|\lambda|^2+|\rho|^2)}\,\varphi_\lambda(a)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,.$$

  - Explicit formula when G/K of complex type: $h_t(a)$ is constant multiple of:

$$(4\pi t)^{-n/2}\,e^{-t|\rho|^2}\prod_{\alpha\in\Sigma^+}\frac{\alpha(\log a)}{2\sinh\alpha(\log a)}\,e^{-|\log a|^2/4t}\,.$$

  - Wind-up over **cocompact** Γ: heat kernel on $\Gamma\backslash G/K$.
- Fay 1977: noncompact quotient of $G/K = \mathbb{H}^d$
- Convergence of $\sum_{\gamma\in\Gamma} h_t(\gamma g)$ in general?

Norms on groups

G, countably based, locally compact, Hausdorf, unimodular topological group G with compact subgroup K

Norm on G, a continuous function $\|\cdot\| : G \to (0, \infty)$ with:

- $\| \mathrm{id}_G \| = 1$, where id_G is the identity element in G,
- $\|g\| \geqslant 1$, for all g in G,
- $\|g\| = \|g^{-1}\|$, for all g in G,
- submultiplicativity: $\|gh\| \leqslant \|g\| \cdot \|h\|$, for all g, h in G,
- K-invariance: $\|kgk'\| = \|g\|$, all g in G, k, k' in K,
- integrability: for some $r_0 \geqslant 0$,

$$\int_G \|g\|^{-r} \, dg < \infty \quad (r > r_0).$$

Garrett's Thm. on Poincaré Series

- G as in previous, Γ discrete subgroup
- Norm $\|\cdot\|$ on G with integrability exponent r_0.
- For suitable $f : G \to \mathbb{C}$, have Poincaré series:

$$\text{Pé}_f(g) \;=\; \sum_{\gamma \in \Gamma} f(\gamma g)$$

Theorem (Garrett; see 2010 paper with Diaconu)

- *If* $|f(g)| \ll \|g\|^{-r}$ *for some* $r > r_0$*, then the associated Poincaré series converges absolutely and uniformly on compact sets to a function of moderate growth.*
- *If* $|f(g)| \ll \|g\|^{-2r}$ *for some* $r > r_0$*, then* Pé_f *is square integrable modulo* Γ*.*

Poincaré series for heat kernel

Theorem (D.)

For $t > 0$, the Poincaré series $\sum_{\gamma \in \Gamma} h_t(\gamma g)$

- *converges absolutely and uniformly on compacts,*
- *is of moderate growth, and*
- *is square integrable mod* Γ*.*

Outline of Proof.

First show:

- $\|g\| = \|kak'\| = e^{|\log a|}$ *is a norm on* G,
- *with integrability expt:* $r_0 = \sum_{\alpha \in \Sigma^+} m_\alpha |\alpha|$.

Debiard, Gaveau, and Mazet bound:

$$h_t(a) \ll (4\pi t)^{-\dim(G/K)/2} \cdot e^{-|\log a|^2/4t},$$

ensures $h_t(a) \ll e^{-2r|\log a|}$, *some* $r > r_0$. □

Challenge 2

Given heat ker. on $\Gamma\backslash G/K$, find automorphic spectral exp'n?

Heuristically obvious, but hard if starting from $\sum h_t(\gamma g)$.

More structural approach (D. 2021)

- Characterize automorphic heat kernel (via afc PDE).
- Construct solution to automorphic PDE.
  - Use global automorphic Sobolev theory.
  - Construct automorphic heat kernel via automorphic spectral expansion in terms of cusp forms, Eisenstein series, and residues of Eisenstein series.

Spectral Theory for $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

Consider $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$, with Laplacian $\Delta = y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$.

Spectral inversion: eigenfunction expansion

$$f \overset{L^2}{=} \sum_F \langle f, F\rangle \cdot F + \langle f, \Phi_0\rangle \cdot \Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \langle f, E_s\rangle \cdot E_s \, ds$$

where

- F in o.n.b. of cusp forms,
- Φ_0 is the constant automorphic form with unit L^2-norm,
- and E_s is the real analytic Eisenstein series.

Note: integrals are extensions by isometric isomorphisms of continuous linear functionals on $C_c^\infty(X)$.

Automorphic Spectral Theory

Abbreviate (and generalize): denote elements of the spectral "basis" (cusp forms, Eisenstein series, residues of Eisenstein series) uniformly as $\{\Phi_\xi\}_{\xi\in\Xi}$.

$$f \;=\; \int_\Xi \langle f, \Phi_\xi\rangle \cdot \Phi_\xi \; d\xi$$

View Ξ as a finite disjoint union of spaces of the form $\mathbb{Z}^n \times \mathbb{R}^m$ with usual measures.

Automorphic Sobolev Spaces

Inner product $\langle\, , \,\rangle_s$ (for $0 \leqslant s \in \mathbb{Z}$) on $C_c^\infty(X)$ by

$$\langle \varphi, \psi \rangle_s = \langle (1 - \Delta)^s \varphi, \psi \rangle_{L^2}$$

Sobolev spaces:

- H^s is Hilbert space completion of $C_c^\infty(X)$ w.r.t. topology induced by $\langle\, , \,\rangle_s$
- H^{-s} is Hilbert space dual of H^s.

Note:

- $H^0 = L^2(X)$
- Nesting: $H^s \hookrightarrow H^{s-1}$ for all s.

Automorphic Heat Kernel

Let ℓ be the smallest integer *strictly greater* than $\dim X/2$.

We define an *automorphic heat kernel* to be a map $U : (0, \infty) \to H^{-\ell}(X)$ such that

1. U satisfies the "initial condition,"

$$\lim_{t \to 0^+} U(t) \;=\; \delta \quad \text{in } H^{-\ell}(X).$$

2. For some $s \leqslant -\ell - 2$, U is strongly differentiable as an H^s-valued function and satisfies the "heat equation", i.e. for $t > 0$,

$$U'(t) - \Delta U(t) \;=\; 0 \quad \text{in } H^s(X)$$

Results from D. 2021

For $t \geqslant 0$, let $u(t) \;=\; \int_\Xi \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t} \cdot \Phi_\xi \, d\xi$.

1. For $t \geqslant 0$, $u(t) \in H^{-\ell}$.
2. $\lim\limits_{t \to 0^+} u(t) \;=\; \delta$ in the topology of $H^{-\ell}$.
3. $u(t)$ is the **unique** automorphic heat kernel.
4. It is strongly C^1 as a $H^{-\ell-2}$-valued function on $[0, \infty)$.
5. For $t > 0$, $u(t)$ lies in $C^\infty(X)$, and its automorphic spectral expansion converges in the $C^\infty(X)$-topology.

Key is to prove analogous results for $\widetilde{u}(t) = \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t}$, which is in a weighted L^2-space on Ξ.

Example: $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

The unique automorphic heat kernel on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$ is:

$$U(t) = \sum_F \overline{F}(x_0)\, e^{\lambda_F t} \cdot F + \overline{\Phi}_0(x_0) \cdot \Phi_0 + \frac{1}{4\pi i} \int_{\frac{1}{2}+i\mathbb{R}} \overline{E}_s(x_0)\, e^{s(s-1)t} \cdot E_s\, ds$$

For $t > 0$, $U(t)$ is a smooth function on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, and its spectral expansion converges to it in the C^∞-topology.

Challenge 4

Jorgenson and Lang proposed a “Gauss” transform:

$$f \;\mapsto\; 2s\int_0^\infty f(t)\,e^{-s^2t}\,dt\,.$$

Also suggested renormalizations and other modifications, e.g.:

$$f \;\mapsto\; 2s\int_0^\infty f(t)\,t^N e^{-s^2t}\,dt\,,$$

for convergence of transform of “trivial” term.

How to choose appropriate transform? What is the effect of applying a Gauss transform, in general? Can we characterize the result?

Again, will shift from formulaic to more structural viewpoint.

Operator and Vector-Valued Transforms

Gauss transform is Laplace transform with change of variables.

- Can apply the robust theory of operator and vector-valued Laplace transform (Arendt, Batty, Hieber, Neubrander).

Given an automorphic heat kernel $u : (0, \infty) \to H^{-\ell-2}$, its Laplace transform,

$$\mathcal{L}u(\lambda) \ = \ \int_0^\infty e^{-\lambda t} \cdot u(t)\, dt$$

exists as an $H^{-\ell-2}$-valued function of λ, holomorphic for λ in a right half plane, and it satisfies

$$(\lambda - \Delta)\mathcal{L}u(\lambda) \ = \ \delta \ ,$$

i.e. $v_\lambda = \mathcal{L}u(\lambda)$ is a fundamental solution for $(\lambda - \Delta)$.

Afc Fundamental Solution

From prior work (D. 2012), we know

- v_λ has an automorphic spectral expansion

$$v_\lambda \;=\; \int_\Xi \frac{\overline{\Phi_\xi(x_0)}}{\lambda - \lambda_\xi}\, \Phi_\xi \, d\xi \quad (\text{in } H^{-\ell+2}),$$

- v_λ is *not* in H^s for any $s \geqslant -\frac{1}{2}(\dim X)$
  - only if $\dim X = 1$ does Sobolev embedding imply v_λ continuous

Smoothed Fundamental Solution

However, if we modify the afc PDE:

$$(\lambda - \Delta)^p \, v_{p,\lambda} \;=\; \delta \,,$$

the solution $v_{p,\lambda}$ has spectral expansion

$$v_\lambda \;=\; \int_\Xi \frac{\overline{\Phi_\xi(x_0)}}{(\lambda - \lambda_\xi)^p} \, \Phi_\xi \, d\xi \quad (\text{in } H^{-\ell+2p}),$$

so for p sufficiently large $v_{p,\lambda}$ is continuous (even C^k).

Connection to Modified Transform

Moreover, we have shown

$$\frac{d^N}{d\lambda^N} v_\lambda = v_{N+1,\lambda} ,$$

and (ABHN, Theorem 1.5.1 $\Rightarrow$),

$$\frac{d^N}{d\lambda^N} v_\lambda = -\mathcal{L}^N U(\lambda) ,$$

where $\mathcal{L}^N$ is the modified Laplace transform:

$$\mathcal{L}^N : f \rightarrow \int_0^\infty e^{-\lambda t} (-t)^N \cdot f(t)\, dt .$$

Therefore,

$$\mathcal{L}^N U(\lambda) = -v_{N+1,\lambda} ,$$

which is continuous (even C^k) for N sufficiently large.

Antidifferentiation Trick?

If $\mathcal{L}(f)$ does not converge,

- Consider $\mathcal{L}^N(f)$ and antidifferentiate with respect to λ.
  - Attempt to recover $\mathcal{L}(f)$.
- For example, with "trivial" term in some spectral expansion.

Caution:

- We know that v_λ is not in H^s for any $s \geqslant -\dim(X)/2$.
- Not reasonable to expect v_λ to have meaningful pointwise values when $\dim(G/K) > 1$.

Mero. Cont'n & Branching

Moreover, (D. 2015; see also Garrett 2018, 12.5),

- Automorphic fundamental solution may exhibit *branching*:
  - Meromorphic continuations along different paths in the complex plane may differ by a term of moderate growth.
  - In particular, the resulting function may lie outside of global automorphic Sobolev spaces.'
- For example:
  - Hilbert-Maass fundamental solutions,
  - GL_3 fundamental solution.
- In general when there is "mixed" discrete/continuous spectrum.

Apparent symmetry which suggests functional equation may be only superficial.

Conclusion: Resolved

- Convergence of geometric description of afc heat kernel and square integrability modulo Γ.
  - Norms on groups, Garrett's theorem, DGM bound.
- Eigenfunction expansion
  - Existence, uniqueness of afc heat kernel as $H^{-\ell}$-valued function of t; in $C^1([0, \infty), H^{-\ell})$; smoothness for $t > 0$.
- Nature of Laplace transform of afc heat kernel (whole).

Conclusion: To Investigate

- Spectral Identity:
  - Does the geometric construction yield a function in $C^1([0,\infty), H^{-\ell})$?
  - Nature of equality of geometric and spectral sides?
- What distribution to apply?
  - Trace? (Automorphic truncation needed?)
  - Evaluation?
- Break up spectral expansion to isolate term of interest?
  - Serendipitous cancellation? (Jo-Lang 2009)
  - Projection to subspace spanned by discrete spectrum.

Thank you for your attention!