The Automorphic Heat Kernel: Spectral and Geometric Points of View

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Applications to Number Theory

- asymptotic formulas for spectra of $\Gamma\backslash G$
  - Gangolli 1968, Donnelly 1982, Deitmar-Hoffman 1999
- relationship between η-invariants and closed geodesics
  - Moscovici-Stanton 1989
- zeta functions from heat Eisenstein series
  - Jorgenson-Lang 1996, 2001, 2008, 2009, 2012
- sup-norm bounds for automorphic forms
  - Jo-Kra04, Jo-Kra11, Ary16, Fr-Jo-Kra16, Ary-Bal18
- limit formulas, Weyl-type asymptotic for period integrals
  - Tsuzuki 2008, 2009
- ave. holo. QUE for afc cfms for quaternion algebras
  - Aryasomayajula-Balasubramanyam 2018

Typical Construction

Wind-up heat kernel on G/K

- Gangolli 1968:
  - integral representation for heat kernel on G/K
  - explicit formula when G/K of complex type
  - wind-up by averaging over cocompact Γ
- Special cases: $G/K = \mathbb{H}^d$, $G = SL_n(\mathbb{C})$, etc.
  - e.g. Fay 1977, Jorgenson-Lang 2009
- Convergence in general? (Existence?)
- Automorphic spectral expansion?
  - "conjectural" (Jorgenson-Lang 2009)

Our Approach

Spectral:

- using global automorphic Sobolev theory
- construct automorphic heat kernel via automorphic spectral expansion in terms of cusp forms, Eisenstein series, and residues of Eisenstein series
  - existence of automorphic heat kernel
- prove uniqueness (semigroup theory)
- prove C^∞-convergence of automorphic spectral expansion and smoothness of automorphic heat kernel (for $t > 0$)

Geometric:

- use known bound on heat kernel on G/K
- wind-up: proof involves norm on G

1D Euclidean Heat Kernel

Heat kernel $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ satisfies:

$$(\partial_t - \Delta)\, u = 0, \qquad \lim_{t \to 0^+} u(x,t) = \delta.$$

Apply Fourier transform $\mathcal{F}$:

$$(\partial_t + 4\pi^2 \xi^2)\, \mathcal{F}u = 0, \qquad \lim_{t \to 0^+} (\mathcal{F}u)(\xi, t) = \mathcal{F}\delta = 1.$$

Considering ξ as fixed, $\mathcal{F}u(\xi, t)$ satisfies familiar IVP:

$$\frac{dy}{dt} = -4\pi^2 \xi^2\, y, \quad y(0) = 1 \quad \Rightarrow \quad y(t) = e^{-4\pi^2 \xi^2 t}$$

Fourier inversion: $u(x,t) = (4\pi t)^{-1/2}\, e^{-x^2/4t}$.

Automorphic Analogue

- $X = \Gamma\backslash G/K$, with G, red. or ss. Lie group, max. compact $K \subset G$, arithmetic $\Gamma \subset G$
- Δ, Laplacian on $\Gamma\backslash G$ (the image of Casimir)
- δ, automorphic delta distribution at $x_0 = \Gamma \cdot 1 \cdot K$

Want $u(x,t)$ on $X \times (0,\infty)$ satisfying

$$(\partial_t - \Delta)\, u = 0 \qquad \text{and} \qquad \lim_{t\to 0^+} u(x,t) = \delta$$

Apply spectral transform $\mathcal{F}$ to get IVP on spectral side

$$(\partial_t - \lambda_\xi)\, \mathcal{F}u = 0 \qquad \text{and} \qquad \lim_{t\to 0^+} \mathcal{F}u(\xi,t) = \mathcal{F}\delta$$

Solve IVP: $\mathcal{F}(u,\xi) = \mathcal{F}\delta \cdot e^{\lambda_\xi t}$; spectral inversion $\to u(x,t)$.

(Global Afc) Sobolev Theory

$$\begin{array}{ccc} \text{Physical Side} & & \text{Spectral Side} \\ X,\ \Delta & \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} & \Xi,\ \lambda_\xi \\ \text{differentiation} & & \text{multiplication} \end{array}$$

Solve differential equations by division!

- Only for Schwartz functions? . . . L^2-functions?
- Functions in (global afc) Sobolev spaces: $H^s(X)$

Other applications:

- lattice point counting in G/K (D. 2012)
- behavior of 4-loop supergraviton (Klinger-Logan, 2018)

Time Parameter

- View heat kernel as H^s-valued function of t.

$$u : (0, \infty) \to H^s(X)$$

- Limit as $t \to 0^+$ is in H^s-topology
- Strong differentiation (vs weak) w.r.t. t
- Translation Lemma
  - from physical side to spectral side and back
  - limits, weak and strong differentiability, differential equations for u and $\mathcal{F} \circ u$

Spectral Theory for $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

Consider $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$, with Laplacian $\Delta = y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$.

Spectral inversion: eigenfunction expansion

$$f \overset{L^2}{=} \sum_F \langle f, F\rangle \cdot F + \langle f, \Phi_0\rangle \cdot \Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \langle f, E_s\rangle \cdot E_s \, ds$$

where

- F in o.n.b. of cusp forms,
- Φ_0 is the constant automorphic form with unit L^2-norm,
- and E_s is the real analytic Eisenstein series

Note: integrals are extensions by isometric isomorphisms of continuous linear functionals on $C_c^\infty(X)$.

Automorphic Spectral Theory

Abbreviate (and generalize): denote elements of the spectral "basis" (cusp forms, Eisenstein series, residues of Eisenstein series) uniformly as $\{\Phi_\xi\}_{\xi \in \Xi}$.

$$f = \int_\Xi \langle f, \Phi_\xi \rangle \cdot \Phi_\xi \, d\xi$$

View Ξ as a finite disjoint union of spaces of the form $\mathbb{Z}^n \times \mathbb{R}^m$ with usual measures.

Automorphic Sobolev Spaces

Inner product $\langle\, , \,\rangle_s$ (for $0 \leqslant s \in \mathbb{Z}$) on $C_c^\infty(X)$ by

$$\langle \varphi, \psi \rangle_s = \langle (1-\Delta)^s \varphi, \psi \rangle_{L^2}$$

Sobolev spaces:

- H^s is Hilbert space completion of $C_c^\infty(X)$ w.r.t. topology induced by $\langle\, , \,\rangle_s$
- H^{-s} is Hilbert space dual of H^s.

Note:

- $H^0 = L^2(X)$
- Nesting: $H^s \hookrightarrow H^{s-1}$ for all s.

Diff'n and Spectral Transform

$$
\begin{array}{ccccccc}
\dots & H^{+s} & \xrightarrow[\approx]{(1-\Delta)} & H^{+s-2} & \xrightarrow[\approx]{(1-\Delta)} & \dots \\
 & \mathcal{F}\downarrow\approx & & \mathcal{F}\downarrow\approx & & \\
\dots & V^{+s} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{+s-2} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & \dots
\end{array}
$$

- spectral transform $\mathcal{F} : f \mapsto \langle f, \Phi_\xi \rangle$
- λ_ξ is the Δ-eigenvalue of Φ_ξ
- weighted L^2-space V^s: $f \in V^s$ means $(1-\lambda_\xi)^{s/2} f \in L^2(\Xi)$

Note:

- Δ nonpositive symmetric operator $\Rightarrow \; \lambda_\xi \leqslant 0$
- $\Lambda : \xi \mapsto \lambda_\xi$ is differentiable and of moderate growth by a pre-trace formula

Key Results

- Every $u \in H^s$ has a spectral expansion, converging in the H^s-topology.
- Global automorphic Sobolev embedding theorem
  - For $s > k + (\dim X)/2$, $H^s \hookrightarrow C^k$.
  - Implies: $H^\infty = C^\infty$
- Pretrace formula $\Rightarrow \delta \in H^s$ for every $s < -(\dim X/2)$

$$\delta \ = \ \int_\Xi \overline{\Phi}_\xi(x_0)\, \Phi_\xi \ d\xi \quad (\text{conv. in } H^s, s < -(\dim X/2))$$

- Expected spectral coefficient for automorphic heat kernel:

$$\mathcal{F}\delta \cdot e^{\lambda_\xi t} \ = \ \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t}.$$

Automorphic Heat Kernel

Let ℓ be the smallest integer *strictly greater* than $\dim X/2$.

We define an *automorphic heat kernel* to be a map $U : (0, \infty) \to H^{-\ell}(X)$ such that

1. U satisfies the "initial condition,"

$$\lim_{t \to 0^+} U(t) \;=\; \delta \quad \text{in } H^{-\ell}(X).$$

2. For some $s \leqslant -\ell - 2$, U is strongly differentiable as an H^s-valued function and satisfies the "heat equation", i.e. for $t > 0$,

$$U'(t) - \Delta U(t) \;=\; 0 \quad \text{in } H^s(X)$$

Existence; Spectral Expansion

For $t \geqslant 0$, let $U(t) \; = \; \int_\Xi \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t} \cdot \Phi_\xi \, d\xi$.

Theorem (1)

1. *For* $t \geqslant 0$, $U(t) \in H^{-\ell}$.
2. $\lim_{t \to 0^+} U(t) \; = \; \delta$ *in the topology of* $H^{-\ell}$.
3. *For* $s \leqslant -\ell - 5$, *viewing* U *as a* H^s*-valued function,* U *is strongly* C^1 *on* $(0, \infty)$ *and satisfies the "heat equation," i.e. for* $t > 0$,

$$\frac{d}{dt}U(t) \; - \; \Delta\, U(t) \;\; = \;\; 0 \quad \textit{in } H^s,$$

where $\frac{d}{dt}U$ *denotes the strong derivative of* U.

In particular, $U(t)$ *is an automorphic heat kernel.*

Proof outline

For $t \geqslant 0$, let $\widetilde{U}(t) : \ \xi \ \mapsto \ \overline{\Phi}_\xi(x_0)\, e^{\lambda_\xi\, t}$.

Prove that:

- $\widetilde{U}$ takes values in $V^{-\ell}$.
- $\widetilde{U}(t) \to \mathcal{F}\delta$ in $V^{-\ell}$ as $t \to 0^+$.
- $\widetilde{U}$ is weakly C^k when viewed as a $V^{-\ell-2N}$-valued function, for $N > k$
  - weakly C^2 when viewed as $V^{-\ell-5}$-valued function
  - strongly C^1 (by weak-to-strong diff. principle)
- $\widetilde{U}$ satisfies the (strong) differential equation

$$\frac{d}{dt} Y(t) = \lambda_\xi Y(t)$$

when viewed as a $V^{-\ell-5}$-valued function.

Use the translation lemma.

Uniqueness and improved differentiability

Theorem (2)

1. *The automorphic heat kernel constructed in Theorem 1 is the* **unique** *automorphic heat kernel.*
2. *It is strongly* C^1 *as a* $\mathrm{H}^{-\ell-2}$*-valued function on* $[0, \infty)$*.*

NB: By Thm 1, $\mathfrak{U}$ is strongly C^1 as a $\mathrm{H}^{-\ell-5}$-valued function.

Idea of proof: use semigroup theory to prove uniqueness of solution to IVP on spectral side.

Proof outline

Prove uniqueness of $\widetilde{U}$ as (suitable) V^s-valued solution to IVP:

$$\frac{d}{dt}Y(t) \ = \ \lambda_\xi \, Y(t), \quad Y(0) = \overline{\Phi}_\xi(x_0)$$

Multiplication by λ_ξ: $V^{s+2} \to V^s$

- continuous linear map when V^{s+2} and V^s have their own (different) topologies as (differently) weighted L^2 spaces
- change perspective: view V^{s+2} as subspace of V^s
  - wreak continuity
  - unbounded operator on Hilbert space V^s
  - prove: densely defined, negative, self-adjoint
  - prove: resolvent set contains $(0, \infty)$
  - infinitesimal generator of a SCCSG (Hille-Yosida)

Abstract Cauchy Problem

Have shown that multiplication by λ_ξ is the infinitesimal generator of a SCCSG, so can use the following:

Proposition

Let $G(t)$ be a SCSG in a Banach space V, let A be the infinitesimal generator for $G(t)$ with domain D, and $v_0 \in D$. Then there is a unique function $[0, \infty) \to V$ that (i) is strongly continuous on $[0, \infty)$, (ii) is strongly differentiable on $(0, \infty)$, (iii) takes values in D, and (iv) solves the initial value problem,

$$\frac{d}{dt} Y(t) \;=\; A\, Y(t)\;; \quad Y(0) \;=\; v_0\,.$$

Moreover, the solution is strongly C^1 on $[0, \infty)$.

Smoothness

Theorem (3)

For $t > 0$, the automorphic heat kernel lies in $C^\infty(X)$, and its automorphic spectral expansion

$$U(t) \;=\; \int_\Xi \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t} \cdot \Phi_\xi \, d\xi$$

converges in the $C^\infty(X)$-topology.

Proof outline.

- *prove:* $\widetilde{U}(t) \in V^s$ *for all s*
- *thus* $U(t) \in H^s(X)$ *for all s*
- *global automorphic Sobolev embedding theorem* $\Rightarrow U(t) \in C^k$ *for all* k

□

Example: $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

Corollary

The unique automorphic heat kernel on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$ is:

$$\begin{aligned} U(t) &= \sum_F \overline{F}(x_0)\, e^{\lambda_F t} \cdot F + \overline{\Phi}_0(x_0) \cdot \Phi_0 \\ &\quad + \frac{1}{4\pi i} \int_{\frac{1}{2}+i\mathbb{R}} \overline{E}_s(x_0)\, e^{s(s-1)t} \cdot E_s \, ds \end{aligned}$$

For $t > 0$, $U(t)$ is a smooth function on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, and its spectral expansion converges to it in the C^∞-topology.

Geometric Perspective

Bi-K-invariant heat kernel on G, conn. ss. Lie, finite center.

- Construct via spherical inversion:

$$h_t(a) = \int_{W\backslash\mathfrak{a}^*} e^{-t(|\lambda|^2+|\rho|^2)} \varphi_\lambda(a)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,.$$

- For G complex, $h_t(a)$ is a constant times:

$$(4\pi t)^{-n/2}\, e^{-t|\rho|^2} \prod_{\alpha\in\Sigma^+} \frac{\alpha(\log a)}{2\sinh\alpha(\log a)}\, e^{-|\log a|^2/4t}\,.$$

For Γ discrete, try winding up: $\sum_{\gamma\in\Gamma} h_t(\gamma g)$.

Convergence? What kind of function on $\Gamma\backslash G/K$?

Geometric Construction

Theorem (4)

For $t > 0$*, the Poincaré series* $\sum_{\gamma \in \Gamma} h_t(\gamma g)$

- *converges absolutely and uniformly on compacts,*
- *is of moderate growth, and*
- *is square integrable mod* Γ*.*

Proof uses:

- norms on groups arguments for convergence etc. (Garrett)
- non-trival (but not sharp) bound for h_t (Anker et al.)

Norms on groups

G, countably based, locally compact, Hausdorf, unimodular topological group G with compact subgroup K

Norm on G, a continuous function $\|\cdot\| : G \to (0, \infty)$ with:

- $\| \mathrm{id}_G \| = 1$, where id_G is the identity element in G,
- $\|g\| \geqslant 1$, for all g in G,
- $\|g\| = \|g^{-1}\|$, for all g in G,
- submultiplicativity: $\|gh\| \leqslant \|g\| \cdot \|h\|$, for all g, h in G,
- K-invariance: $\|kgk'\| = \|g\|$, all g in G, k, k' in K,
- integrability: for some $r_0 \geqslant 0$,

$$\int_G \|g\|^{-r} \, dg < \infty \quad (r > r_0).$$

Poincaré series

- G as in previous, Γ discrete subgroup
- Norm $\|\cdot\|$ on G with integrability exponent r_0.
- For suitable $f : G \to \mathbb{C}$, have Poincaré series:

$$\text{Pé}_f(g) \;=\; \sum_{\gamma \in \Gamma} f(\gamma g)$$

Theorem (Garrett; see 2010 paper with Diaconu)

- *If* $|f(g)| \ll \|g\|^{-r}$ *for some* $r > r_0$, *then the associated Poincaré series converges absolutely and uniformly on compact sets to a function of moderate growth.*
- *If* $|f(g)| \ll \|g\|^{-2r}$ *for some* $r > r_0$, *then* Pé_f *is square integrable modulo* Γ*.*

Outline of proof of Theorem 4

Prove:

- $\|g\| = \|kak'\| = e^{|\log a|}$ is a norm on G,
- with integrability expt: $r_0 = \sum_{\alpha \in \Sigma^+} m_\alpha |\alpha|$.

To apply Garrett's theorem, want:

$$h_t(a) \ll e^{-2r|\log a|}, \quad \text{some } r > r_0.$$

Non-trivial (but not sharp) bound (Anker et. al) suffices:

$$h_t(a) \ll t^{-n/2} \, e^{-|\rho|^2 t - \langle \rho, \log a \rangle - |\log a|^2/4t} \quad (t > 0).$$

Further

Poincaré series gives a **weak** automorphic heat kernel.

- Poincaré series is in $L^2(X) = H^0(X) \subset H^{-\ell}(X)$.
- Limit as $t \to 0^+$ approaches δ weakly in $H^{-\ell}$.
- Weakly differentiable as H^s-valued function of t and satisfies weak version of automorphic heat equation.

Is the Poincaré series an automorphic heat kernel, as we have defined it? **If so**, can apply uniqueness theorem:

- it has the automorphic spectral expansion stated earlier,
- converging in the C^∞ topology for $(t > 0)$,
- so for $t > 0$ is a smooth function on X.

Thank you for your attention!