Global Automorphic Sobolev Theory and Automorphic Heat Kernels

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Automorphic Heat Kernels

Applications to Number Theory:

- Weyl Law (Müller, 2007)
- periods of wave functions (Tsuzuki 2009)
- integral representations for Selberg zeta functions (Jorgenson and Lang 2009)
- sup norm bounds for Bergman kernels (e.g. Bouche 1996, Berman 2004, Jorgenson and Kramer 2004, Aryasomayajula 2016)
- etc.

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Typically: wind-up free-space heat kernel

- explicit formulas for heat kernels on symmetric spaces of complex type (Gangolli)
- automorphic spectral expansion?

Our Approach

Solve automorphic PDE:

- use spectral theory of automorphic forms
- construct solution via automorphic spectral expansion in terms of cusp forms, Eisenstein series, and residues of Eisenstein series
- framework of global automorphic Sobolev theory gives clear conclusions about convergence

1D Euclidean Heat Kernel

Heat equation:

$$(\partial_t - \Delta)\, u \;=\; 0$$

Heat kernel is fundamental solution, i.e. satisfies

$$\lim_{t \to 0} u(x,t) \;=\; \delta$$

Use Fourier transform and Fourier inversion to derive

$$u(x,t) \;=\; \frac{e^{-x^2/4t}}{2\sqrt{\pi t}} \qquad x \in \mathbb{R},\; t > 0$$

Heuristic Derivation

Apply Fourier transform $\mathcal{F}$ to heat equation:

$$(\partial_t - \Delta)\, u \;=\; 0 \quad \Rightarrow \quad (\partial_t + 4\pi^2\xi^2)\, \mathcal{F}u \;=\; 0$$

since (d.u.t.i.s and i.b.p)

$$\mathcal{F}(\partial_t u) \;=\; \partial_t(\mathcal{F}u) \quad \text{and} \quad \mathcal{F}(\Delta u) \;=\; -4\pi^2\xi^2(\mathcal{F}u)$$

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Considering ξ as fixed, $\mathcal{F}u(\xi, t)$ satisfies familiar ODE

$$\frac{dy}{dt} = -4\pi^2\xi^2\, y$$

so, for some C_ξ independent of t,

$$(\mathcal{F}u)(\xi, t) = C_\xi e^{-4\pi^2\xi^2 t}$$

Heuristic Derivation (con't)

Apply Fourier transform $\mathcal{F}$ to initial condition:

$$\lim_{t\to 0} u(x,t) \;=\; \delta \quad\Rightarrow\quad \lim_{t\to 0}(\mathcal{F}u)(\xi,t) \;=\; \mathcal{F}\delta \;=\; 1$$

But $(\mathcal{F}u)(\xi,t) = C_\xi e^{-4\pi^2\xi^2 t}$, so $C_\xi = 1$.

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Fourier inversion:

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} e^{-4\pi^2\xi^2 t} \cdot e^{2\pi i x \xi}\, d\xi \\ &= \int_{\mathbb{R}} e^{-\pi(4\pi t\, \xi^2 - 2ix\,\xi)}\, d\xi \\ &= \frac{e^{-x^2/4t}}{2\sqrt{\pi t}} \end{aligned}$$

Further Details Needed

- $\lim_{t \to 0} u(x,t) = \delta$ in what space of functions of x?
- Fix one variable, and let the other vary?
- Extend Fourier transform beyond L^2? to apply e.g. to δ?
- u is "nice enough"?

Set-up and Heuristic

- G is a reductive or semi-simple Lie group
- with discrete subgroup Γ
- and maximal compact subgroup K
- $X = \Gamma\backslash G/K$
- Δ is the Laplacian on $\Gamma\backslash G$ (the image of Casimir)
- δ is the automorphic delta distribution at $x_0 = \Gamma \cdot 1 \cdot K$

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Want to construct $u(x,t)$ on $X \times (0,\infty)$ satisfying

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...suitably interpreted.

Strategy (roughly)

Apply a spectral transform in the spatial variable to:

- both sides of the heat equation
- both sides of the initial condition

Need a framework broad enough to apply the spectral transform to the delta distribution

- not a test function (nor a Schwartz function)
- not even an element of L^2

Global automorphic Sobolev theory (D. 2011) allows us to treat the spectral transform, inversion, and differentiation in a rigorous and robust setting.

Spectral Theory for $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

Consider $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$, with Laplacian $\Delta = y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$.

Spectral inversion: eigenfunction expansion

$$f \overset{L^2}{=} \sum_F \langle f, F\rangle \cdot F + \langle f, \Phi_0\rangle \cdot \Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \langle f, E_s\rangle \cdot E_s \, ds$$

where

- F in o.n.b. of cusp forms,
- Φ_0 is the constant automorphic form with unit L^2-norm,
- and E_s is the real analytic Eisenstein series

Note: integrals are extensions by isometric isomorphisms of continuous linear functionals on $C_c^\infty(X)$.

Automorphic Spectral Theory

Abbreviate (and generalize): denote elements of the spectral "basis" (cusp forms, Eisenstein series, residues of Eisenstein series) uniformly as $\{\Phi_\xi\}_{\xi \in \Xi}$.

$$f = \int_\Xi \langle f, \Phi_\xi \rangle \cdot \Phi_\xi \, d\xi$$

View Ξ as a disjoint union of Euclidean spaces with the counting measure on each copy of $\mathbb{R}^0$ and the usual Euclidean measure on each copy of $\mathbb{R}^n$.

Automorphic Sobolev Spaces

Inner product $\langle\, , \rangle_s$ (for $0 < s \in \mathbb{Z}$) on $C_c^\infty(X)$ by

$$\langle \varphi, \psi \rangle_s = \langle (1-\Delta)^s \varphi, \psi \rangle_{L^2}$$

Sobolev spaces:

- H^s is Hilbert space completion of $C_c^\infty(X)$ w.r.t. topology induced by $\langle\, , \rangle_s$
- H^{-s} is Hilbert space dual of H^s.

Note:

- $H^0 = L^2(X)$
- Nesting: $H^s \hookrightarrow H^{s-1}$ for all s.

Diff'n and Spectral Transform

$$\begin{array}{ccccccc} \ldots & H^{+s} & \xrightarrow[\approx]{(1-\Delta)} & H^{+s-2} & \xrightarrow[\approx]{(1-\Delta)} & \ldots \\ & \mathcal{F}\downarrow\approx & & \mathcal{F}\downarrow\approx & & \\ \ldots & V^{+s} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{+s-2} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & \ldots \end{array}$$

- spectral transform $\mathcal{F} : f \mapsto \langle f, \Phi_\xi \rangle$
- λ_ξ is the Δ-eigenvalue of Φ_ξ
- weighted L^2-space V^s: $f \in V^s$ means $(1-\lambda_\xi)^{s/2} f \in L^2(\Xi)$

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Note:

- Δ nonpositive symmetric operator $\Rightarrow \lambda_\xi \geqslant 0$
- $\Lambda : \xi \mapsto \lambda_\xi$ is differentiable and of moderate growth

Key Results

- Every $u \in H^s$ has a spectral expansion, converging in the H^s-topology.
- Global automorphic Sobolev embedding theorem
  - For $s > k + (\dim X)/2$, $H^s \hookrightarrow C^k$.
  - Implies: $H^\infty = C^\infty$
- Pretrace formula $\Rightarrow \delta \in H^s$ for every $s < -(\dim X/2)$
- So

$$\delta = \int_\Xi \overline{\Phi}_\xi(x_0)\, \Phi_\xi \, d\xi \qquad (\text{conv. in } H^s, s < -(\dim X/2))$$

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1. U is strongly differentiable on $(0,\infty)$, i.e. for $t > 0$,

$$U'(t) \;=\; \lim_{h\to 0} \frac{U(t+h) - U(t)}{h} \qquad \text{exists in } H^{-\ell}(X)$$

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$$\lim_{t \to 0} U(t) = \delta \qquad \text{in } H^{-\ell}(X)$$

3. U satisfies the heat equation, i.e. for all $t > 0$,

$$U'(t) - \Delta U(t) = 0 \qquad \text{in } H^{-\ell-2}(X)$$

Theorem

For $t > 0$, the following automorphic spectral expansion

$$U(t) \;=\; \int_{\Xi} \overline{\Phi}_{\xi}(x_0) \cdot e^{\lambda_{\xi} t} \cdot \Phi_{\xi} \, d\xi$$

converges with respect to *all* global automorphic Sobolev topologies, and thus converges in $C^{\infty}(X)$. For $t = 0$ the expansion converges in the $H^{-\ell}$-topology to the automorphic delta distribution. Thus this expansion defines an automorphic heat kernel.

Corollary

In the case of $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$, we have the following automorphic spectral expansion for the automorphic heat kernel,

$$U(t) = \sum_F \overline{F}(x_0)\, e^{\lambda_F t} \cdot F + \overline{\Phi}_0(x_0) \cdot \Phi_0 + \frac{1}{4\pi i} \int_{\frac{1}{2}+i\mathbb{R}} \overline{E}_s(x_0)\, e^{s(s-1)t} \cdot E_s \, ds$$

For $t > 0$, the spectral expansion converges in the C^∞-topology (so, in particular, uniformly pointwise) to a smooth function on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, and for $t = 0$, it converges to the automorphic delta distribution.

The Spectral Side

Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ are isometric isomorphisms $H^{-\ell} \leftrightarrow V^{-\ell}$,

- strong diff. of $u \iff$ strong diff. of $\mathcal{F} \circ u$
- u satisfies the heat equation

$$u'(t) \;=\; \Delta u(t) \;\text{ in } H^{-\ell}$$

if and only if $\mathcal{F} \circ u$ satisfies the "eigenfunction equation"

$$(\mathcal{F} \circ u)'(t) \;=\; \Lambda \otimes (\mathcal{F} \circ u)(t) \;\text{ in } V^{-\ell}$$

where $\Lambda : \Xi \to \mathbb{R}$ by $\Lambda(\xi) = \lambda_\xi$

Proposed Spectral Coefficients

For each $t \geqslant 0$, let $\widetilde{U}(t)$ be the map

$$\widetilde{U}(t) : \xi \mapsto \overline{\Phi}_\xi(x_0) \cdot e^{\lambda_\xi t} = \overline{\Phi}_\xi(x_0) \cdot e^{-|\lambda_\xi| t}$$

Then $\widetilde{U}$ as a $V^{-\ell}$-valued function of t, since

- $\xi \mapsto \overline{\Phi}_\xi(x_0)$ lies in $V^{-\ell}$, and
- $\xi \mapsto e^{-|\lambda_\xi| t}$ is continuous and bounded for $t \geqslant 0$

With $\widetilde{U}(t) : \Xi \to V^{-\ell}$ by $\xi \mapsto \overline{\Phi}_\xi(x_0) \cdot e^{-|\lambda_\xi|\,t}$, as above,

- Weak-to-strong smoothness: $\widetilde{U}$ is strongly differentiable, as a $V^{-\ell}$-valued function of t
- Hahn-Banach theorem:

$$\widetilde{U}'(t) \;=\; \Lambda \otimes \widetilde{U}(t)$$

- for $t > 0$, for all s, $(1 + |\lambda_\xi|)^{s/2} e^{-|\lambda_\xi|t}$ is continuous and bounded so, for $t > 0$, $\widetilde{U}(t)$ lies in V^∞.

Back to Physical Side

Now let $U(t) = \mathcal{F}^{-1} \circ \widetilde{U}$, i.e.

$$U(t) = \int_{\Xi} \overline{\Phi}_{\xi}(x_0) \cdot e^{-|\lambda_{\xi}|\,t} \cdot \Phi_{\xi}\, d\xi$$

By the corresponding properties for $\widetilde{U}$, we can conclude

- U strongly diff. $H^{-\ell}$-valued function
- U satisfies the heat equation
- $U(t)$ is in fact smooth for $t > 0$

Uniqueness

Spectral transform of an automorphic heat kernel satisfies

$$\frac{d}{dt}Y \;=\; \Lambda \otimes Y$$

where Y is a $V^{-\ell}$-valued function of t.

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Ansatz: general solution is $v \otimes \mathcal{E}$, where

- v is a function in $V^{-\ell}$, not depending on t, and
- $\mathcal{E}$ is the (function-on-Ξ)-valued function of t given by

$$\mathcal{E}(t) : \xi \mapsto e^{\lambda_\xi t}$$

Then $\widetilde{U}$ would be the unique solution satisfying the initial condition. Apply inverse spectral transform: uniqueness of automorphic heat kernel.

Thank you for your attention!