Applications of Modern Analysis to Automorphic Forms and Analytic Number Theory

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Overview

Applications of automorphic differential equations:

- subconvexity (growth of ζ/L-functions on critical line)
- lattice point counting in symmetric spaces (analogue of Gauss circle problem)
- vanishing of ζ/L-functions on critical line

Underlying framework:

- Automorphic spectral expansions
- Global automorphic Sobolev theory (to make heuristic "engineering math" arguments into rigorous ones with clear conclusions)

The Riemann Zeta Function

RH: Nontrivial zeros of ζ (probably) all lie on $\frac{1}{2} + i\mathbb{R}$.

Implied growth of ζ on $\frac{1}{2} + i\mathbb{R}$:

$$|\zeta(\tfrac{1}{2} + it)| \ll_\varepsilon (1 + |t|)^\varepsilon \;\; \forall\, \varepsilon > 0 \qquad \text{(LH)}$$

Equivalent to optimal error term in PNT.

Convexity ("trivial") bound:

$$|\zeta(\tfrac{1}{2} + it)| \ll_\varepsilon (1 + |t|)^{\frac{1}{4} + \varepsilon} \;\; \forall\, \varepsilon > 0$$

Subconvexity: Reduce $\frac{1}{4}$ in exponent. E.g. Weyl, 1921: $\frac{1}{6}$.

Other ζ/L-functions

Analogous conjectures and results for other ζ/L-functions.

NB: Interesting applications follow already from subconvexity.

- Cogdell 2003 (with Piatetski-Shapiro, Sarnak): representability of algebraic integer as sum of three squares in a ring of algebraic integers
- Watson 2002: QUE for arithmetic surfaces would follow from subconvex bounds for certain degree eight L-functions

Moreover, for some automorphic L-functions, even the "trivial" convexity bound has not been proven

Poincaré Series for Subconvexity

Diaconu-Garrett, 2010: subconvexity for GL_2 automorphic L-function over arbitrary number field in t-aspect

Poincaré series whose kernel is neither smooth nor compactly supported

- solution to differential equation $(\Delta - \lambda) u = \theta_H$ on symmetric space G/K
- θ_H is distribution: integrate along subgroup H

Poincaré series itself is solution to corresponding automorphic differential equation

- heuristically immediate automorphic spectral expansion

On Higher Rank Groups

D. 2016: Constructing Poincaré series

- Explicit formulas when G is a complex semi-simple Lie group
- Harmonic analysis on symmetric spaces
- Global zonal spherical Sobolev spaces
- Poincaré series produce identities involving moments of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ Rankin-Selberg L-functions

Lattice Point Counting

Classical Gauss Circle Problem:

- Elementary packing arguments:

$$N(T) \;=\; \#\{\xi \in \mathbb{Z}^2 : |\xi| \leqslant T\} \;=\; \pi \cdot T^2 \;+\; O(T)$$

- Optimal error term conjectured to be $O(T^{1/2+\varepsilon})$.

Asymptotic in non-Euclidean spaces?

- Hyperbolic spaces: subtler than packing arguments
- Affine symmetric spaces: ergodic methods

Symmetric space G/K, G complex semisimple Lie group

- Poincaré series from automorphic differential equation

$$(\Delta - \lambda)u \;=\; \delta$$

- exact formula relating number of lattice points with automorphic spectrum (D. 2012)

RH and Eigenvalues

Hilbert-Polya: To prove RH, construct a self-adjoint operator on a Hilbert space with eigenvalues $\lambda_s = s(s-1)$ parametrized by zeros of the Riemann zeta function.

Intriguing investigations, 1977-1983:

- Haas, 1977: List of parameters for (purported) eigenvalues of Δ on the modular curve $SL_2(\mathbb{Z})\backslash\mathfrak{H}$.
- Stark noticed zeros of zeta.
- Hejhal, Colin de Verdière, 1981-1983: reproduce? repair?

Bombieri-Garrett, preprint, *Designed Pseudo-Laplacians*

Possible Repair

Haas' error: overlooking non-smoothness of (purported) eigenfunctions at corner ω of fundamental domain

Repair: construct operator (pseudo-Laplacian) "$\widetilde{\Delta}_\omega$" that overlooks this non-smoothness?

- Haas' spurious eigenvalues of Δ would be genuine eigenvalues of $\widetilde{\Delta}_\omega$?
- Would have self-adjoint operator on a Hilbert space some of whose eigenvalues are parametrized by zeros of zeta on the critical line?
- Would want to show that all (or at least many) of the zeros of ζ are accounted for in this way ...

Reason for Hope

In Colin de Verdière's meromorphic continuation of Eisenstein series:

- Certain truncated Eisenstein series (not smooth!) are genuine eigenfunctions for a pseudo-Laplacian $\widetilde{\Delta}_a$.
- For u in the domain of $\widetilde{\Delta}_a$,

$$(\widetilde{\Delta}_a - \lambda)u = 0 \iff (\Delta - \lambda)u = (\text{const.})\,\eta_a$$

  where η_a is the distribution that evaluates the constant term at height a.

Imitate this to construct self-adjoint $\widetilde{\Delta}_\omega$ such that

$$(\widetilde{\Delta}_\omega - \lambda)u = 0 \overset{\text{(naive hope)}}{\iff} (\Delta - \lambda)u = (\text{const.})\,\delta_\omega$$

where δ_ω is the automorphic Dirac delta at $\omega = e^{\pi i/3}$?

Imitating CdV

With

- Δ_ω, restriction of Δ to $C_c^\infty(X) \cap \ker(\delta_\omega)$,
- $\widetilde{\Delta}_\omega$, Friedrichs extension of Δ_ω (canonical, self-adjoint),

Then, indeed, for u **in the domain of** $\widetilde{\Delta}_\omega$,

$$(\widetilde{\Delta}_\omega - \lambda)u = 0 \iff (\Delta - \lambda)u = (\text{const.})\,\delta_\omega$$

And, using global automorphic Sobolev theory, can construct solutions to distributional differential equation using automorphic spectral expansions.

Failure of Naive Repair

Again, we have: for u **in the domain of** $\widetilde{\Delta}_\omega$,

$$(\widetilde{\Delta}_\omega - \lambda)u \;=\; 0 \quad \Longleftrightarrow \quad (\Delta - \lambda)u \;=\; (\text{const.})\,\delta_\omega$$

However, global automorphic Sobolev theory also shows that

- **No solution will lie in the domain of** $\widetilde{\Delta}_\omega$.
- Discrete spectrum of $\widetilde{\Delta}_\omega$ is **empty**.

CdV's suggestion: replace δ by θ, a projection of δ to the non-cuspidal part of the automorphic spectrum.

Bombieri-Garrett prove: existence of $\widetilde{\Delta}_\theta$-eigenvalue $\lambda_w = w(w-1)$ with $\text{Re}(w) = \frac{1}{2}$ does imply vanishing of zeta function at w.

Constructing the Pseudo-Laplacian

Let Θ be a compactly supported distribution on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$ and θ the projection to the non-cuspidal part of the spectrum.

Restrict Δ to $L^2_{nc}(X) \cap C_c^\infty(X) \cap \ker(\theta)$, and let $\widetilde{\Delta}_\theta$ be its Friedrichs extension.

Then, for u in the domain of $\widetilde{\Delta}_\theta$,

$$(\widetilde{\Delta}_\theta - \lambda_w)u = 0 \qquad \Longleftrightarrow \qquad (\Delta - \lambda_w)u = (\text{const}) \cdot \theta$$

Vanishing of Periods

Theorem (Bombieri-Garrett)

Let θ and $\widetilde{\Delta}_\theta$ be as above. Suppose θ lies in $H^{-1}(X)$ and θ is real, in the sense that $\theta(\overline{\varphi}) = \overline{\theta(\varphi)}$ for all $\varphi \in C_c^\infty(X)$. Then the compact period θE_w vanishes when $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$.

Note

Hardy-Littlewood 1918 $\Rightarrow \theta = \mathrm{Proj}_{\mathrm{nc}}\delta_\omega$ satisfies the hypotheses. Here: $\theta E_s = E_s(\omega) = (\sqrt{3}/2)^s \zeta_{\mathbb{Q}(\omega)}(s)/\zeta(2s)$.

Corollary

Let $\theta = \mathrm{Proj}_{\mathrm{nc}}\delta_\omega$. If $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$, then $\zeta_{\mathbb{Q}(\omega)}(w) = 0$.

How Many Zeros?

With $\lambda_w = w(w-1)$ an eigenvalue for $\widetilde{\Delta}_\theta$,

Have shown:

- w's on the critical line $\subset$ zeros of zeta function

Hope: w's account for

- **all** zeros of zeta? (No: Epstein zetas.)
- **many** zeros of zeta? (Not clear.)

If Montgomery's Pair Correlation Conjecture is true,

- at most a **positive fraction** of zeros of zeta,
- because they **interlace** with zeros of $c_P(E_s)(ia) = a^s + c_s a^{1-s}$ on the critical line.

GL_3 Automorphic L-functions

- GL_3 automorphic L-functions arise as compact periods of GL_3 cuspidal data Eisenstein series. (Lapid et. al.)
- Theorem (D.) Vanishing of compact periods of GL_3 cuspidal data Eisenstein series at w-values on the critical line corresponding to eigenvalues (if any) of suitable pseudo-Laplacian.
- Theorem (D.) Interlacing with discrete spectrum of pseudo-Laplacian on Lax-Phillips space.
- H^{-1}-condition on period distribution: condition on the second moment of period.
- Given suitable moment bound, prove vanishing of GL_3 automorphic L-functions?

Periods of Totally Degenerate Eisenstein Series

For a degree n extension ℓ of a number field k, let H be the copy of $\ell^\times$ in $GL_n(k)$. Then

$$\Theta_H(E_s^{\deg}) = \int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s^{\deg}(h)\, dh = \frac{\xi_\ell(s)}{\xi_k(ns)}$$

where $E_s^{\deg}$ is a totally degenerate GL_n Eisenstein series.

Prove vanishing of ξ_ℓ on the critical line?

Problem: Totally degenerate Eisenstein series do not occur in automorphic spectral expansion.

Tension: The easier it is to prove that a period has an Euler product, the less likely it is that the Eisenstein series occurs in the automorphic spectral expansion.

Thank you for your attention!