Branching of Automorphic Fundamental Solutions

Amy DeCelles

University of St. Thomas

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Outline

Solutions of Automorphic Differential Equations

- Poincaré series of Good, Diaconu-Goldfeld, Diaconu-Garrett
    - subconvexity of GL_2 automorphic L-functions in the t aspect over an arbitrary number field
- higher rank: Diaconu-Garrett moment identities
- lattice point counting in symmetric spaces G/K where G is complex (D)
- eigenfunctions for pseduo-differential operators
  . . . meromorphic continuation of Eisenstein series (CdV)

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Fundamental Solutions

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Here we are interested in fundamental solutions for $(\Delta - \lambda)$ on $\Gamma\backslash G/K$ where

- G is a reductive or semi-simple Lie group, $K \subset G$ maximal compact, $\Gamma \subset G$ discrete
- Δ is the Laplacian, the image of Casimir for $\mathfrak{g}$
- λ is a complex parameter
- δ is Dirac delta at basepoint

Engineering Math

Spectral expansions immediate (heuristically).

Example: Fourier expansions

$$\begin{aligned}
\left(\frac{d^2}{dx^2} - 4\pi^2 w^2\right) u_w &= \delta \\
\mathcal{F}\left(\left(\frac{d^2}{dx^2} - 4\pi^2 w^2\right) u_w\right) &= \mathcal{F}(\delta) \\
-4\pi^2(\xi^2 + w^2)\mathcal{F}u_w &= 1
\end{aligned}$$

Thus $\mathcal{F}u_w = -1/(4\pi^2(\xi^2 + w^2))$, and

$$u_w = \int_{-\infty}^{\infty} \frac{-e^{2\pi i x\xi}\, d\xi}{4\pi^2(\xi^2 + w^2)} = \frac{-e^{2\pi w|x|}}{4\pi w} \qquad (\mathrm{Re}(w) > 0)$$

Similarly . . .

Simplest automorphic case: $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

$$u_w = \sum_f \frac{\bar{f}(z_0)\, f}{\lambda_f - \lambda_w} + \frac{\bar{\Phi}_0(z_0)\, \Phi_0}{\lambda_0 - \lambda_w}$$

$$+ \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{E_{\frac{1}{2}-it}(z_0)\, E_{\frac{1}{2}+it}\, dt}{\lambda_s - \lambda_w} \qquad (\mathrm{Re}(w) > \tfrac{1}{2})$$

where z_0 is the base point in $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, f ranges over an orthonormal basis of cusp forms, Φ_0 is the constant automorphic form, $s = \frac{1}{2} + it$, $\lambda_w = w(w-1)$, and λ_f, λ_0, and λ_s are the eigenvalues of f, Φ_0, and E_s, respectively.

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Convergence?? Global automorphic Sobolev theory! (D)

The Geometric Side

Poincaré series: wind-up corresponding fundamental solution on free space G/K.

$$\mathrm{Pé}_w(g) \;=\; \sum_{\gamma\in\Gamma} u_w^{\mathrm{free}}(\gamma g)$$

Then $\mathrm{Pé}_w$ is an automorphic fundamental solution.

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Then Pé_w is an automorphic fundamental solution.

Perhaps more common to start with Pé_w, find its spectral expansion, hoping for meromorphic continuation in w.

However: sometimes the automorphic fundamental solution exhibits branching!

Outline

To be more precise . . .

$$(\Delta - \lambda_w)^\nu \, u_w \;=\; \delta_{z_o} \qquad (\nu \in \mathbb{N})$$

Global automorphic Sobolev theory ensures:

- solution u_w exists and is unique in global automorphic Sobolev spaces
- automorphic spectral expansion converges in Sobolev topology for $\mathrm{Re}(w) \gg 1$
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However:

- meromorphic continuations along different w-paths may differ by a term of moderate growth (branching)
- the resulting function may lie outside of global automorphic Sobolev spaces

Hilbert-Maass Fundamental Solutions

Let k be a totally real number field of degree $n > 1$ over $\mathbb{Q}$, and let $\mathfrak{o}$ be its ring of integers.

For $\mathrm{Re}(w) > \frac{1}{2}$, there is a unique solution u_w to the automorphic differential equation $(\Delta - \lambda_w)u_w = \delta$:

$$u_w = \sum_F \frac{\overline{F}(z_o) \cdot F}{\lambda_F - \lambda_w} + \frac{1}{(\lambda_1 - \lambda_w)\langle 1, 1\rangle} + \sum_\chi \frac{1}{4\pi i} \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s,\overline{\chi}}(z_o) \cdot E_{s,\chi}}{\lambda_{s,\chi} - \lambda_w}\, ds$$

For each nontrivial unramified grossencharacter χ, the corresponding integral $\mathcal{I}_\chi$ has *two branch points* on the critical line.

Fix a grossencharacter χ. Let $\sigma_1, \ldots \sigma_n$ be the archimedean places of k. Take real parameters $t_\chi = (t_1, \ldots, t_n)$ with $t_1 + \cdots + t_n = 0$ such that

$$\chi(\alpha) = \sigma(\alpha)^{it_1} \ldots \sigma_n(\alpha)^{it_n}$$

where $\alpha \in (k \otimes_{\mathbb{Q}} \mathbb{R})^\times$, and let

$$\|t_\chi\|^2 = \frac{1}{n} \left(|t_1|^2 + \ldots + |t_n|^2\right)$$

Writing the eigenvalue in terms of s and t_χ,

$$\lambda_{s,\chi} = \frac{1}{n} \Big((s + it_1)(s + it_1 - 1) + \ldots + (s + it_n)(s + it_n - 1)\Big)$$

Thus the integrand has poles at

$$s = \tfrac{1}{2} \pm \sqrt{(w - \tfrac{1}{2})^2 + \|t_\chi\|^2}$$

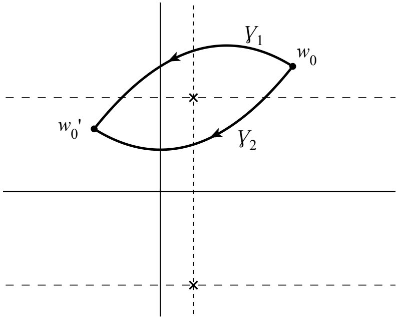

Figure: Pathwise meromorphic continuation along these two paths in the w-plane yields functions that differ by a term of moderate growth. The dotted vertical line is the critical line $\mathrm{Re}(w) = \frac{1}{2}$. The dashed horizontal lines are $\mathrm{Im}(w) = \pm\|t_\chi\|$.

Theorem

Let γ_1 and γ_2 be w-paths in $\mathbb{C}$, each originating at a point w_0 in the right half plane $\mathrm{Re}(w) > \frac{1}{2}$, crossing the critical line once, and terminating at a point w_0' in the left half plane $\mathrm{Re}(w) < \frac{1}{2}$, with γ_1 crossing the critical line at a height greater than $\|t_\chi\|$ and γ_2 crossing at a height less than $\|t_\chi\|$. Then pathwise meromorphic continuations of $\mathcal{I}_\chi(w)$ along the paths γ_1 and γ_2 differ by a term of moderate growth, namely by

$$\frac{4\pi i \;\cdot\; E_{1-s(\chi,w),\overline{\chi}}(z_o) \;\cdot\; E_{s(\chi,w),\chi}}{1-2s(\chi,w)}$$

where, $s(\chi,w)$ is defined as follows. For fixed w in $\mathrm{Re}(s) > \frac{1}{2}$, $s(\chi,w)$ is the pole of the integrand of $\mathcal{I}_\chi(w)$ in $\mathrm{Re}(s) > \frac{1}{2}$. As w crosses the critical line, $s(\chi,w)$ is defined by analytic continuation.

We sketch the proof.

Regularize:

$$\mathcal{J}_\chi(w) \;=\; \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s,\overline{\chi}}(z_o)\,E_{s,\chi} \;-\; E_{1-s(\chi,w),\overline{\chi}}(z_o)\,E_{s(\chi,w),\chi}}{\lambda_{s,\chi}-\lambda_w}\,ds$$
$$+\; E_{1-s(\chi,w),\overline{\chi}}(z_o)\,E_{s(\chi,w),\chi} \;\cdot\; \int_{\frac{1}{2}+i\mathbb{R}} \frac{ds}{\lambda_{s,\chi}-\lambda_w}$$

By design the integrand of the first integral on the right side is continuous. The second integral can be evaluated by residues:

$$2\pi i \times \operatorname*{Res}_{s\,=\,1-s(\chi,w)} \frac{1}{(s-s(\chi,w))(s-(1-s(\chi,w)))} \;=\; \frac{2\pi i}{1-2s(\chi,w)}$$

Consider $\chi = 1$. Then $s(\chi, w) = s(1, w) = w$, and

$$\mathcal{I}_1(w) = \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s,1}(z_o)\, E_{s,1} - E_{1-w,1}(z_o)\, E_{w,1}}{\lambda_{s,1} - \lambda_w}\, ds$$

$$+ \; E_{1-w,1}(z_o)\, E_{w,1} \cdot \frac{2\pi i}{1-2w} \qquad (\mathrm{Re}(w) > \tfrac{1}{2})$$

Move w across the critical line and reverse the regularization:

$$\mathcal{I}_1(w) = \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s,1}(z_o)\, E_{s,1}}{\lambda_{s,1} - \lambda_w}\, ds$$

$$- \; E_{1-w,1}(z_o)\, E_{w,1} \times$$

$$\left(\int_{\frac{1}{2}+i\mathbb{R}} \frac{1}{\lambda_{s,1} - \lambda_w}\, ds - \frac{2\pi i}{1-2w} \right) (\mathrm{Re}(w) < \tfrac{1}{2})$$

Since $s = w$ is now the pole to the left of the critical line, residue calculus yields

$$\int_{\frac{1}{2}+i\mathbb{R}} \frac{ds}{\lambda_{s,\chi} - \lambda_w} = 2\pi i \times \operatorname*{Res}_{s=w} \frac{1}{(s-w)(s-(1-w))} = \frac{2\pi i}{2w-1}$$

Thus the integral corresponding to $\chi = 1$ is

$$\mathcal{I}_1(w) = \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s,1}(z_o)\, E_{s,1}}{\lambda_{s,1} - \lambda_w}\, ds \qquad (\mathrm{Re}(w) > \tfrac{1}{2})$$

$$\mathcal{I}_1(w) = \int_{\frac{1}{2}+i\mathbb{R}} \frac{E_{1-s,1}(z_o)\, E_{s,1}}{\lambda_{s,1} - \lambda_w}\, ds + E_{1-w,1}(z_o)\, E_{w,1} \cdot \frac{4\pi i}{1-2w} \qquad (\mathrm{Re}(w) < \tfrac{1}{2})$$

Thus we see that the pathwise meromorphic continuation has an **additional term** when w is left of the critical line.

Now: χ nontrivial.

- If w crosses the critical line with imaginary part greater in magnitude than $\|t_\chi\|$, the radicand, $(w - \frac{1}{2})^2 + \|t_\chi\|^2$, in the expression for $s(\chi, w)$ moves around the branch point of the square root, the origin. As above, get **additional term**.

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- If w crosses the critical line with imaginary part within a distance of $\|t_\chi\|$ of the real axis, the radicand, $(w-\frac{1}{2})^2 + \|t_\chi\|^2$, stays strictly in the right half plane and thus does *not* travel around the origin. **No additional term**.

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- If w crosses the critical line with imaginary part within a distance of $\|t_\chi\|$ of the real axis, the radicand, $(w-\frac{1}{2})^2 + \|t_\chi\|^2$, stays strictly in the right half plane and thus does *not* travel around the origin. **No additional term**.

Thus **branching** is evident: pathwise meromorphic continuations of $\mathcal{I}_\chi(w)$ depend non-trivially on the path, the branch points being $w = \frac{1}{2} \pm i\,\|t_\chi\|$.

GL_3 Automorphic Fundamental Solution

Let $G = SL_3(\mathbb{R})$, $K = SO(3)$ and $\Gamma = SL_3(\mathbb{Z})$.

$$u_w = \sum_{\text{cfm } F} \frac{\overline{F}(x_0)}{(\lambda_F - \lambda_w)^\nu} \cdot F + \frac{1}{\langle 1, 1\rangle (\lambda_1 - \lambda_w)^\nu}$$

$$+ \frac{1}{|W|} \int_{\rho + i\mathfrak{a}^*} \frac{E_{\bar{\chi}_\mu}(x_0)}{(\lambda_\chi - \lambda_w)^\nu} \cdot E_{\chi_\mu} \, d\mu$$

$$+ \sum_{GL_2 \text{ cfms } f} \int_{\frac{1}{2} + i\mathbb{R}} \frac{E_{\bar{f}, 1-s}(x_0)}{(\lambda_{f,s} - \lambda_w)^\nu} \cdot E_{f,s} \, ds$$

For each GL_2 cusp form f in the chosen orthonormal basis, the corresponding integral has *two branch points* on the critical line.

Acknowledgements

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For a preprint, see arXiv:1401.2015 [math.NT] or visit

http://personal.stthomas.edu/dece4515