Bruhat trick: spherical function as an integral over affine

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Another approach for determining the spherical functions for GL_3. Characterize spherical function as left average over K of a spherical vector in principal series. Use Bruhat decomposition to write as an integral over $\overline{N}$ (affine!) This is very rough. *At several points I relied on Helgason's 1984 book.*

0. Context

Previously, we had characterized the spherical function as the (unique) bi-K-invariant eigenfunction for $\mathfrak{z}$ ($\approx$ the algebra of left G-invariant differential operators on G/K, for classical groups at least) satisfying a normalization condition $\varphi(1) = 1$ and perhaps also a growth condition (tempered?) To determine the spherical functions on $SL_2(\mathbb{C})$, we can (by hand) compute the Casimir operator in radial coordinates and find a (constant coefficient!) DE

$$(\Omega - \lambda)\,\frac{\varphi}{\sinh r} = 0 \quad \iff \quad \varphi'' + ((\text{const}) - \lambda) \times \varphi = 0$$

With a little work, we get

$$\varphi_s = \frac{\sinh((2s-1)r)}{(2s-1)\sinh r}$$

For $SL_3(\mathbb{C})$, spherical functions are still be elementary, but imitating the procedure that worked for $SL_2(\mathbb{C})$ turns out to be troublesome. Although we can compute Casimir on bi-K-invariant functions and get a constant coefficient PDE, this PDE is not sufficient to determine the spherical function, because we need to use the action of the *other* generator of the center of the universal enveloping algebra, which is a differential operator of order four, and which does *not* have a tractable explicit description.

As an alternative, we characterize the spherical function as the average of left K-translates of the spherical vector in the principal series,

$$\varphi_s(g) = \int_K f_s(kg)\,dg \qquad f_s \in I_s = \text{Ind}_P^G(\chi_s)$$

First we treat the case of $SL_2(\mathbb{R})$ as the very simplest example (though the integral is non-elementary–we'll get a Bessel function) then $SL_2(\mathbb{C})$, using a formula for the Haar measure on $SU(2)$ lifted from Helgason, and finally the case of $SL_3(\mathbb{C})$. Along the way we'll discuss computation of the Haar measure of SL_3 in Iwasawa coordinates, the Bruhat decomposition of GL_3, and the change of measure $\bar{N} \to K/M$.

1. The Cases $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$

Consider the case of $SL_2(\mathbb{R})$ first. The integral we want to compute is

$$\int_K f_s(kg)\,dk = \int_{SO(2)} f_s(kg)\,dk = \int_0^{2\pi} f_s(k_\theta g)\,d\theta$$

We use the facts that f_s is right K-invariant and left P-equivariant by χ_s. The integral is left and right K-invariant, so we may assume that $g \in A^+$.

$$kg = ka_r = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{r/2} & \\ & e^{-r/2} \end{pmatrix} = \begin{pmatrix} e^{r/2}\cos\theta & e^{r/2}\sin\theta \\ -e^{r/2}\sin\theta & e^{-r/2}\cos\theta \end{pmatrix}$$

We can right multiply by a suitable element of K to get the argument in P.

$$ka_rk' = \begin{pmatrix} * & * \\ 0 & \sqrt{(-e^{-r}\sin\theta)^2 + (e^r\cos\theta)^2} \end{pmatrix}$$

So the integral becomes

$$\int_0^{2\pi} ((-e^r \sin\theta)^2 + (e^{-r}\cos\theta)^2)^{-s/2}\, d\theta$$

This integral is non-elementary: a hypergeometric function. (good)

We might try doing the same thing for $SL_2(\mathbb{C})$ and $SU(2)$. In particular, we'd want to right-multiply an arbitrary ka_r by some k' to get something in P, use the left P-equivariance of f_s, and integrate over $SU(2)$. However, the Haar measure on $SU(2)$ is not as simple as the Haar measure on $SO(2) \approx S^1$, so this is not as trivial.

For the present discussion, we will write $U = SU(2)$ (instead of K) since we will want to use K to refer to a subgroup of $SU(2)$. We write

$$u \in SU(2) \text{ is } u = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$$

In Cartan coordinates,

$$u = k_\theta a_t k_\phi \quad \text{where } k_\theta = \begin{pmatrix} e^{(i/2)\theta} & \\ & e^{-(i/2)\theta} \end{pmatrix},\ a_t = \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{pmatrix}$$

The Haar measure of $SU(2)$ in Cartan coordinates is given by

$$\int_{SU(2)} f(u)\, du = \frac{1}{16\pi^2} \int_0^\pi \sin t\, dt \int_0^{2\pi} d\theta \int_{-2\pi}^{2\pi} f(k_\theta, a_t k_\phi)\, d\phi$$

(I found this in Hegason, exercise 11 of Ch I.)

The spherical function for $SL_2(\mathbb{C})$ is given by the integral

$$\int_{SU(2)} f_s(ug)\, du = \int_{SU(2)} f_s(ua_r)\, du$$

We would like the argument of f_s to be in P.

For an arbitrary $g \in G$, we can right-multiply by an element u' of $SU(2)$ to get something in P.

$$gu' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

A little computation shows that we must have

$$\begin{pmatrix} \bar\beta/c & * \\ 0 & c/\bar\beta \end{pmatrix}$$

If we pick β such that

$$|\beta|^2 = \frac{|c|^2}{|c|^2 + |d|^2}$$

then

$$\chi_s(gu') = |\bar\beta/c|^{4s} = (|c|^2 + |d|^2)^{2s}$$

For $g = ua_r$, $u \in SU(2)$, a_r in the standard maximal torus of G,

$$ua_r = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \begin{pmatrix} e^{r/2} & \\ & e^{-r/2} \end{pmatrix} = \begin{pmatrix} \alpha e^{r/2} & \beta e^{-r/2} \\ -\bar\beta e^{r/2} & \bar\alpha e^{-r/2} \end{pmatrix}$$

Right multiplying by a suitable $u' \in SU(2)$ and applying χ_s,

$$\chi_s(ua_r u') = (|\beta|^2 e^r + |\alpha|^2 e^{-r})^{2s}$$

Putting u into Cartan coordinates, this is

$$\chi_s(ua_r u') = (\sin^2(t/2)e^r + \cos^2(t/2)e^{-r})^{2s}$$

So the integral is

$$\begin{aligned}
\int_{SU(2)} f_s(ua_r)\,du &= \int_{SU(2)} f_s(ua_r u')\,du \\
&= \int_{SU(2)} \chi_s(ua_r u')\,du \\
&= \frac{1}{16\pi^2}\int_0^{\pi} \sin t\,dt \int_0^{2\pi} d\theta \int_{-2\pi}^{2\pi} (\sin^2(t/2)e^r + \cos^2(t/2)e^{-r})^{2s}\,d\phi \\
&= \frac{1}{2\pi}\int_0^{\pi} (\sin^2(t/2)e^r + \cos^2(t/2)e^{-r})^{2s}\ \sin t\,dt
\end{aligned}$$

Evaluating the integral yields

$$\frac{\sinh((2s+1)r)}{2^{2s+1}\pi(2s+1)\sinh r}$$

So this is (up to some normalization issues) the same as the expression for the spherical function that we obtained by solving the DE!

2. The case of $SL_3(\mathbb{C})$

Notice, in the $SL_2(\mathbb{C})$ case above, integrating over $K = SU(2)$ is not trivial. We needed to parametrize $SU(2)$ by three real parameters. In order to do this for $SU(3) \subset SL_3(\mathbb{C})$ we would expect nine (eight?) real parameters! So instead of integrating over K, we will integrate over $\overline{N}$, which is *affine*. (Heuristically, passing from K to $\overline{N}$ is like removing a point from a circle, $S^1 \approx SO(2)$, to get a line, $\mathbb{R} \approx \overline{N} \subset SL_2(\mathbb{R})$.)

We are trying to get the spherical function for $SL_3(\mathbb{C})$ by integrating (averaging on the left) the spherical vector in the principal series over K

$$\varphi_s = \int_K f_s(k*)\,dk$$

Since f_s is left P-equivariant, we can write this as an integral over the quotient $(M \cap K)\backslash K$, where M is the Levi component of P

$$\int_{(M\cap K)\backslash K} f_s(\bar{k}*)\,d\bar{k}$$

If $k = m\bar{k} \in M \cdot \big((M \cap K)\backslash K\big)$,

$$\int_K f_s(ka)\,dk = \int_{M\cap K}\int_{(M\cap K)\backslash K} f(m\bar{k}a)\,d\bar{k}\,dm = \int_{M\cap K} \chi_s(m)\,dm \cdot \int_{(M\cap K)\backslash K} f(\bar{k}a)\,d\bar{k}$$

Now M is the subgroup of diagonal matrices and K is $SU(3)$, so $M \cap K$ is a torus

$$M \cap K = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}, \quad |m_1|^2 = |m_2|^2 = |m_3|^2 = 1$$

So $\chi_s(m) = 1$ for $m \in M \cap K$, and the first integral is just the volume of $M \cap K$, which is some finite constant, independent of s. So we just need to compute the second integral

$$\int_{(M\cap K)\backslash K} f(\bar{k}a)\,d\bar{k}$$

This quotient is isomorphic to $P\backslash G$, since

$$(M \cap K)\backslash K = (P \cap K)\backslash K \approx P\backslash PK \approx P\backslash G$$

The Bruhat decomposition for the minimal parabolic,

$$G = \bigsqcup_{w \in W} PwN$$

gives a decomposition of the quotient space $P\backslash G$,

$$P\backslash G = \bigsqcup_{w \in W} P\backslash PwN$$

For $w = w_o$, the long Weyl element,

$$P\backslash PwN \approx P\backslash Pw^{-1}Nw = P\backslash P\overline{N} \approx \overline{N}$$

This PwN, the "big cell," is an open dense subset of G. So there is a map from $\overline{N}$ onto an open dense subset of $P\backslash G$, $\bar{n} \to P \cdot (1 \cdot \kappa(\bar{n}))$, where κ denotes the map, which extracts the K-part of an element of G, according to the Iwasawa decomposition.

(For the next part, we follow Helgason, I.5.3, "Integral Formulas for the Bruhat Decomposition.")

We have a map of $\overline{N}$ onto an open dense subset of $G/P \approx K/M$, by $\bar{n} \to \kappa(\bar{n}) \cdot M$. We want to determine the way the integral transforms

$$\int_{K/M} f(k)\, dk = \int_{\overline{N}} (f \circ \kappa)(\bar{n})\, \psi(\bar{n})\, d\bar{n}$$

To do this we left multiply by an element x of $\overline{N}$ and compute the change of measure on the left hand side.

We can compute the change of measure from basic manipulations with the Haar measure in Iwasawa (KAN) coordinates.

$$\int dg = \int \delta(a)\, dk\, da\, dn$$

This measure is invariant under left translations by arbitrary elements of G, i.e.

$$\int f(xkan)\, \delta(a)\, dk\, da\, dn = \int f(xg)\, dg = \int f(g)\, dg = \int f(kan)\, \delta(a)\, dk\, da\, dn$$

If we then write xk in Iwasawa coordinates: $xk = K(xk) \cdot A(xk) \cdot N(xk)$, we can rewrite the argument

$$xkan = K(xk)\, A(xk)\, N(xk)\, an = K(xk)\, A(xk)\, (aa^{-1})\, N(xk)\, an = K(xk) \cdot A(xk)\, a \cdot (a^{-1}\, N(xk)a)n$$

So the K-part of $xkan$ is $K(xk)$, the A-part is $A(xk) \cdot a$, and the N-part is $(a^{-1}N(xk)a) \cdot n$.

Note. Writing $K(g)$, $A(g)$, and $N(g)$ for the K-part, A-part, and N-part of g is not at all standard. It's a temporary notation. Helgason writes $k(g)$ for the K-part of g, but I thought that was confusing, since k also refers to an arbitrary element of K. It is standard to write $H(g)$ for the $\log_{\mathfrak{a}}$ of the A-part. So Helgason writes $e^{H(g)}$ instead of $A(g)$. For the rest of this discussion I will write

$$\kappa(g) = K\text{-part of } g \quad \text{and} \quad A(g) = A\text{-part of } g$$

Note. In particular, notice that we have shown

$$\kappa(xg) = \kappa(x \cdot \kappa(g))$$

for any x, g in G.

Since da and dn are left Haar measures on A and N, we can change variables $a \to A(xk) \cdot a$ and $n \to (a^{-1}N(xk)a) \cdot n$, without a change of measure. So the integral is

$$\begin{aligned} \int f(xkan)\, \delta(a)\, dk\, da\, dn &= \int f(\kappa(xk)an)\, \delta(a \cdot A(xk)^{-1})\, dk\, da\, dn \\ &= \int f(\kappa(xk)an)\, \delta(a) \cdot \delta(A(xk))^{-1}\, dk\, da\, dn \end{aligned}$$

So we have rewritten the integral of $f(xg) = f(xkan)$, and we are able to conclude, by the left-invariance of $dg = \delta(a)\, dk\, da\, dn$,

$$\int f(kan)\, \delta(a)\, dk\, da\, dn = \int f(\kappa(xk)an)\, \delta(a) \cdot \delta(A(xk))^{-1}\, dk\, da\, dn$$

From this we can see how an interval over K behaves under left G-translation. If F is a function on K,

$$\int_K F(\kappa(xk))\, dk = \int_K F(\kappa(xk))\, \delta(A(xk))^{-1}\, dk$$

To see this just choose f to be a product, $f(kan) = F(k) \cdot F_1(a) \cdot F_2(n)$.

Recall, we are trying to determine the change of measure ψ in

$$\int_{K/M} f(k)\, dk = \int_{\overline{N}} (f \circ \kappa)(\bar{n})\, \psi(\bar{n})\, d\bar{n}$$

We perform a change of variables $\bar{n} \to x\bar{n}$, for some $x \in \overline{N}$. Since $k = \kappa(\bar{n})$, this change of variables sends k to $\kappa(x\bar{n})$. Recall from above that $\kappa(xg) = \kappa(x \cdot \kappa(g))$ for any $g \in G$, so in particular for $g = \bar{n}$. So in terms of k, the change of variables is $k \to \kappa(xk)$.

$$\int_{K/M} f(\kappa(xk))\, dk = \int_{\overline{N}} (f \circ \kappa)(x\bar{n})\, \psi(x\bar{n})\, d\bar{n}$$

Looking at the interval over K, this is precisely the change of variables we discussed above.

$$\int_{K/M} f(\kappa(xk))\, dk = \int_{K/M} f(\kappa(xk))\, \delta(A(xk))^{-1}\, dk$$

Now transform this integral to an integral over $\overline{N}$. Let $g(k) = f(\kappa(xk))\, \delta(A(xk))^{-1}$.

$$\begin{aligned} \int_{K/M} g(k)\, dk &= \int_{\overline{N}} (g \circ \kappa)(\bar{n})\, \psi(\bar{n})\, d\bar{n} \\ &= \int_{\overline{N}} f(\kappa(x \cdot \kappa(\bar{n})))\, \delta(A(x\kappa(\bar{n})))^{-1}\, \psi(\bar{n})\, d\bar{n} \\ &= \int_{\overline{N}} f(\kappa(x\bar{n}))\, \delta(A(x\kappa(\bar{n})))^{-1}\, \psi(\bar{n})\, d\bar{n} \end{aligned}$$

since $\kappa(x \cdot \kappa(\bar{n})) = \kappa(x\bar{n})$. Now putting the two back together

$$\int_{\overline{N}} f(\kappa(x\bar{n}))\, \delta(A(x\kappa(\bar{n})))^{-1}\, \psi(\bar{n})\, d\bar{n} = \int_{\overline{N}} (f \circ \kappa)(x\bar{n})\, \psi(x\bar{n})\, d\bar{n}$$

and $\psi(x\bar{n}) = \delta(A(x\kappa(\bar{n})))^{-1}\, \psi(\bar{n})$. Letting $\bar{n} = 1$, we see $\psi(x) = \delta(A(x))^{-1}$. So the change of measure is

$$\int_{K/M} f(k)\, dk = \int_{\overline{N}} (f \circ \kappa)(\bar{n})\, \delta(A(\bar{n}))^{-1}\, d\bar{n}$$

Unfortunately (from the point of view of explicitly computing the spherical function) we would still need to know how to find the A-part of an arbitrary $\bar{n}$ in order to compute the integral.

A.1 Appendix: Haar measure in Iwasawa coordinates

The aim of this section is to recap the derivation of the Haar measure in Iwasawa coordinates. (See Helgason I.5.1)

$$\int_G f(g)dg = \int_{KAN} f(kan)\,\delta(a)\,dk\,da\,dn \quad \text{where } \delta(a) = e^{2\rho\log a}$$

Since $K \times A \times N \to KAN$ is a diffeomorphism we know that there is a function D such that

$$\int f(g)\,dg = \int f(kan)\,d(kan) = \int f(kan)\,D(k,a,n)\,dk\,da\,dn$$

Since G, K, A, N are all unimodular, G being reductive/semi-simple, K being compact, A and N being abelian,

$$\int f(kan)\,D(k,a,n)\,dk\,da\,n = \int f(g)\,dg = \int f(k'gn')\,dg = \int f(k'k\cdot a\cdot nn')\,D(k'k,a,nn')\,dk\,da\,dn$$

So D only depends on a. Let $\delta(a) = D(k,a,n)$.

For $a' \in A$,

$$\int f(ga')\,dg = \int f(kana')\,\delta(a)\,dk\,da\,dn = \int f(kaa'n')\,\delta(a)\,dk\,da\,dn$$

where $n' = (a')^{-1}na'$. Change variables $a \to a(a')^{-1}$.

$$\int f(kan')\,\delta(a)\,\delta(a')^{-1}\,dk\,da\,dn$$

So we have,

$$\int f(kan)\,\delta(a)\,dk\,da\,dn = \int f(g)\,dg = \int f(ga')\,dg = \int f(kan')\,\delta(a)\,\delta(a')\,dk\,da\,dn$$

To determine δ we need to determine the way that dn transforms under the change of variables $n \to n'$, i.e. under conjugation by elements of A.

It is a general fact that, for oriented manifolds M_1 and M_2, for ω an i-form on M_1, and for Φ an orientation-preserving diffeomorphism,

$$\int_{M_2} f\Phi^*\omega = \int_{M_1} (f\circ\Phi^{-1})\omega$$

for all test functions f, where Φ^* is the transform/pullback by Φ. For Riemannian manifolds, M_1 and M_2 with Riemannian measures dp and dq,

$$\int_{M_2} F(q)\,dq = \int_{M_1} F(\Phi(p))\,|\det d\Phi_p|\,dp$$

Here $M_1 = M_2 = N$, $\Phi(a)$ is the automorphism $n \to ana^{-1}$, and the differential $d\Phi(a)_e : \mathfrak{n}_+ \to \mathfrak{n}_+$ is $(\mathrm{Ad}(a)|\mathfrak{n}_+)$.

To see this we compute $\Phi^*(a) = |\det(d\Phi(a)_e)|$ for $a \in A$, where

$$a = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

Take derivatives of Φ with respect to a basis corresponding to the positive roots: e.g., for GL_3, x_α, x_β, $x_{\alpha+\beta}$.

$$\frac{d}{dt}\Big|_{t=0} \Phi(a)(\exp(tx_\alpha)) = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & a_1a_2^{-1}t & \\ & 1 & \\ & & 1 \end{pmatrix} = a_1a_2^{-1}$$

Then

$$\det(d\Phi(a)_e) = \det \begin{pmatrix} a_1 a_2^{-1} & & \\ & a_2 a_3^{-1} & \\ & & a_1 a_3^{-1} \end{pmatrix} = \left(\frac{a_1}{a_3}\right)^2$$

Notice that this is the same as $\mathrm{Ad}(a)$ on n_+

$$\begin{aligned} \mathrm{Ad}(a)x_\alpha &= a_1 a_2^{-1} \cdot x_\alpha \\ \mathrm{Ad}(a)x_\beta &= a_2 a_3^{-1} \cdot x_\beta \\ \mathrm{Ad}(a)x_{\alpha+\beta} &= a_1 a_3^{-1} \cdot x_{\alpha+\beta} \end{aligned}$$

Putting this in terms of the positive roots,

$$\mathrm{Ad}(a)x_\gamma = e^{\gamma(\log a)} \cdot x_\gamma$$

So $d\Phi(a)_e = (\mathrm{Ad}(a)|\mathfrak{n}_+)$ and

$$\det\left(\mathrm{Ad}\, a|\mathfrak{n}_+\right) = \det \begin{pmatrix} e^{\alpha(\log a)} & & \\ & e^{\beta(\log a)} & \\ & & e^{(\alpha+\beta)(\log a)} \end{pmatrix} = \prod_{\alpha \in \Sigma^+} e^{\alpha(\log a)} = \exp\left(\sum_{\alpha \in \Sigma^+} \alpha(\log a)\right) = e^{2\rho(\log a)}$$

where ρ is half the sum of positive roots. This is true in general, not just for GL_3. The change of measure when $n \to ana^{-1}$ is

$$\Phi^*(a) = \det\left(\mathrm{d}\Phi(a)_e\right) = \det\left(\mathrm{Ad}\, a|\mathfrak{n}_+\right) = e^{2\rho(\log a)}$$

So we have shown that, for all $a' \in A$,

$$\int f(kan)\,\delta(a)\,dk\,da\,dn = \int f(kan)\,\delta(a)\,\delta(a')^{-1}\,e^{2\rho \log a'}\,dk\,da\,dn$$

which implies that $\delta(a') = e^{2\rho \log a'}$, which is what we set out to prove.

A.2 Appendix Haar measure on $SU(2)$

I found a formula for the Haar measure on $SU(2)$ in Cartan coordinates in Helgason (exercise 11, Ch I). I haven't had time to derive the formula myself yet . . .