Notes on Jorgenson and Lang's §1.5, "Characters on the parabolics"

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Explicating and disambiguating
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We explicate and clarify Jorgenson and Lang's discussion in §1.5 of their book on heat Eisenstein series on $SL_n(\mathbb{C})$. We have found a few of their statements to be ambiguous and difficult to interpret, and we suspect that there may be some misprints, as some of the statements are not literally correct. These notes attempt to address the ambiguities and inaccuracies.

1 Preliminaries

We recall some definitions and other preliminary material necessary for stating the results of §1.5 in Jorgenson and Lang's book on heat Eisenstein series on $SL_n(\mathbb{C})$. Besides preceding sections in this book, see also Jorgenson and Lang's book on spherical inversion on $SL_n(\mathbb{R})$. Note that Jorgenson and Lang use some non-standard notation and terminology, due to their preference for explicit and direct arguments rather than making use of more general results in Lie theory.

We start with some definitions from §1.4. Let $G = SL_n(\mathbb{C})$ and $K = SU(n)$, which is a maximal compact subgroup. Let U and A be the subgroups of G consisting of matrices of the following forms:

$$U \ : \ \begin{pmatrix} 1 & & * \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} , \qquad A \ : \ \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} , \quad a_i > 0 \, , \quad \text{so} \ \prod_{i=1}^{n} a_i \ = \ 1.$$

For $n \geq 2$, let $\mathcal{P}$ be a partition of n, i.e. an ordered tuple $\mathcal{P} = (n_1, \ldots, n_{r+1})$ of positive integers whose sum is n. Define non-negative integers m_k, $0 \leq k \leq r+1$ by:

$$m_0 = 0 \, , \quad m_k \ = \ n_1 \, + \, \ldots \, + \, n_k \, , \ \ 1 \leq i \leq r+1 \, .$$

Note that Jorgenson and Lang define m_k for $1 \leq k \leq r+1$; we extend the definition to $k = 0$ to simplify the statement of certain formulas.

Let $U_{\mathcal{P}}$ and $A_{\mathcal{P}}$ denote the subgroups of G consisting of matrices of the following forms:

$$U_{\mathcal{P}} \ : \ \begin{pmatrix} \boxed{\mathbb{1}_{n_1}} & & & * \\ & \boxed{\mathbb{1}_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathbb{1}_{n_{r+1}}} \end{pmatrix} , \qquad A_{\mathcal{P}} \ : \ \begin{pmatrix} \boxed{a_1 \mathbb{1}_{n_1}} & & & 0 \\ & \boxed{a_2 \mathbb{1}_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{a_{n_{r+1}} \mathbb{1}_{n_{r+1}}} \end{pmatrix} , \quad \begin{array}{l} a_i > 0 \text{ for } 1 \leq i \leq r+1, \\ \text{so} \ \prod_{i=1}^{r+1} a_i^{n_i} \ = \ 1. \end{array}$$

Let $G_{\mathcal{P}}$ be the subgroup of block diagonal matrices, in which each block has determinant one, i.e.

$$G_{\mathcal{P}} \ : \ \begin{pmatrix} \boxed{SL_{n_1}(\mathbb{C})} & & & 0 \\ & \boxed{SL_{n_2}(\mathbb{C})} & & \\ & & \ddots & \\ 0 & & & \boxed{SL_{n_{r+1}}(\mathbb{C})} \end{pmatrix} \quad \cong \quad SL_{n_1}(\mathbb{C}) \times \cdots \times SL_{n_{r+1}}(\mathbb{C}) \ .$$

Note that $A_{\mathcal{P}}$ and $G_{\mathcal{P}}$ centralize each other and

$$A_{\mathcal{P}} G_{\mathcal{P}} \cong GL_{n_1}(\mathbb{C}) \times \dots GL_{n_{r+1}}(\mathbb{C}).$$

Further, both $A_{\mathcal{P}}$ and $G_{\mathcal{P}}$ normalize $U_{\mathcal{P}}$. Thus the product $P = U_{\mathcal{P}} A_{\mathcal{P}} G_{\mathcal{P}}$ is a subgroup of G; it is called the standard reduced parabolic subgroup of G corresponding to the partition $\mathcal{P}$.

Let $K_{\mathcal{P}} = SU(n_1) \times \dots SU(n_{r+1}) = K_{G_{\mathcal{P}}}$. Note that $K_{\mathcal{P}} = K \cap P$. See the Remark in §1.4, page 27.

Given a standard reduced parabolic subgroup P of $G = SL_n(\mathbb{C})$ corresponding to a partition $\mathcal{P}$ of n, we may denote the subgroups $U_{\mathcal{P}}$, $A_{\mathcal{P}}$, $G_{\mathcal{P}}$, and $K_{\mathcal{P}}$ by U_P, A_P, G_P, and K_P.

Now proceeding to §1.5, we introduce subgroups U_{G_P} and A_{G_P} of G_P:

$$U_{G_P} \ : \ \begin{pmatrix} \boxed{\begin{matrix} 1 & & * \\ & 1 & \\ 0 & \ddots & 1 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} 1 & & * \\ & 1 & \\ 0 & \ddots & 1 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} 1 & & * \\ & 1 & \\ 0 & \ddots & 1 \end{matrix}} \end{pmatrix} , \qquad A_{G_P} \ : \ \begin{pmatrix} \boxed{\begin{matrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_{m_1} \end{matrix}} & & 0 \\ & \ddots & \\ 0 & & \boxed{\begin{matrix} a_{m_r+1} & & 0 \\ & \ddots & \\ 0 & & a_{m_{r+1}} \end{matrix}} \end{pmatrix} ,$$

where $a_1, \dots, a_{m_{r+1}}$ are positive real numbers with $\prod_{j=m_{k-1}+1}^{m_k} a_j = 1$ for all $1 \le k \le r+1$. Note that Jorgenson and Lang do not explicitly describe these groups.

Now consider the Lie subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ corresponding to the subgroups U and A of $SL_n(\mathbb{C})$:

$$\mathfrak{n} \ = \ \mathrm{Lie}(U) \ : \ \begin{pmatrix} 0 & & * \\ & 0 & \\ 0 & \ddots & 0 \end{pmatrix} , \qquad \mathfrak{a} \ : \ \mathrm{Lie}(A) \ = \ \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ 0 & \ddots & h_n \end{pmatrix} , \ h_i \in \mathbb{R} \, , \ \sum_{i=1}^{n} h_i \ = \ 0.$$

Lie subalgebras of $\mathfrak{n}$ include:

$$\mathfrak{n}_P \;=\; \mathrm{Lie}(U_P)\;:\; \begin{pmatrix} \boxed{\mathbb{0}_{n_1}} & & & * \\ & \boxed{\mathbb{0}_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathbb{0}_{n_{r+1}}} \end{pmatrix}, \qquad \mathfrak{n}_{G_P} \;=\; \mathrm{Lie}(U_{G_P})\;:\; \begin{pmatrix} \boxed{\begin{smallmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{smallmatrix}} & & & 0 \\ & \boxed{\begin{smallmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{smallmatrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{smallmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{smallmatrix}} \end{pmatrix}.$$

And Lie subalgebras of $\mathfrak{a}$ include $\mathfrak{a}_P$, which is constant on each block:

$$\mathfrak{a}_P \;=\; \mathrm{Lie}(A_P)\;:\; \begin{pmatrix} \boxed{h_1 \mathbb{1}_{n_1}} & & & 0 \\ & \boxed{h_2 \mathbb{1}_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{h_{n_{r+1}} \mathbb{1}_{n_{r+1}}} \end{pmatrix}, \qquad \sum_{k=1}^{r+1} n_k h_k = 0\,,$$

and $\mathfrak{a}_{G_P}$, whose blocks each have trace zero:

$$\mathfrak{a}_{G_P} \;=\; \mathrm{Lie}(A_{G_P})\;:\; \begin{pmatrix} \boxed{\begin{smallmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_{m_1} \end{smallmatrix}} & & 0 \\ & \ddots & \\ 0 & & \boxed{\begin{smallmatrix} h_{m_r+1} & & 0 \\ & \ddots & \\ 0 & & h_{m_{r+1}} \end{smallmatrix}} \end{pmatrix}, \qquad \text{such that, for all } 1 \le k \le r+1, \quad \sum_{\ell=1}^{n_k} h_{m_{k-1}+\ell} = 0.$$

Consider the Lie subalgebra corresponding to G_P:

$$\mathfrak{g}_{G_P} \;=\; \mathrm{Lie}(G_P)\;:\; \begin{pmatrix} \boxed{\mathfrak{sl}_{n_1}(\mathbb{C})} & & & 0 \\ & \boxed{\mathfrak{sl}_{n_2}(\mathbb{C})} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathfrak{sl}_{n_{r+1}}(\mathbb{C})} \end{pmatrix} \quad\cong\quad \mathfrak{sl}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_{n_{r+1}}(\mathbb{C})\,,$$

which contains $\mathfrak{n}_{G_P}$ and $\mathfrak{a}_{G_P}$ as well as $\mathfrak{k}_{G_P} = \mathrm{Lie}(K_P) = \mathfrak{su}(n_1) \oplus \cdots \oplus \mathfrak{su}(n_{r+1})$.

Note that $\mathfrak{g}_{G_P}$ has a decomposition $\mathfrak{g}_{G_P} = \mathfrak{n}_{G_P} \oplus \mathfrak{a}_{G_P} \oplus \mathfrak{k}_{G_P}$ and, since $\mathfrak{g}_{G_P} \cong \mathfrak{sl}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_{n_{r+1}}(\mathbb{C})$, each of the Lie subalgebras $\mathfrak{n}_{G_P}$, $\mathfrak{a}_{G_P}$, $\mathfrak{k}_{G_P}$ decomposes into a direct sum of corresponding Lie subalgebras of $\mathfrak{sl}_{n_1}(\mathbb{C})$, ..., $\mathfrak{sl}_{n_{r+1}}(\mathbb{C})$.

2 Characters of $\mathfrak{a}$

Corresponding to the Iwasawa decomposition $G = UAK$, we have what Jorgenson and Lang call "$(\mathfrak{a}, \mathfrak{n})$-relevant characters," which are more commonly called the positive roots. Jorgenson and Lang denote the set of such characters by $\mathcal{R}(\mathfrak{n})$. Explicitly,

$$\mathcal{R}(\mathfrak{n}) \;=\; \left\{ \alpha_{i,j} \in \mathfrak{a}^\vee : \; \alpha_{i,j} \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_i - h_j \text{ for some } 1 \le i < j \le n \right\} ,$$

and it is defined by the fact that:

$$\mathfrak{n} \;=\; \bigoplus_{\alpha \in \mathcal{R}(\mathfrak{n})} \mathfrak{n}_\alpha \,, \quad \text{where } \mathfrak{n}_{\alpha_{i,j}} = \mathbb{C} \cdot E_{i,j} \;.$$

Jorgenson and Lang then introduce $\mathcal{S}(\mathfrak{n})$, the subset of "simple characters" (usually called simple roots) explicitly:

$$\mathcal{S}(\mathfrak{n}) \;=\; \{\alpha_i = \alpha_{i,i+1} \text{ for some } 1 \le i \le n-1\} \;.$$

Note that every $\alpha \in \mathcal{R}(\mathfrak{n})$ is a sum of elements of $\mathcal{S}(\mathfrak{n})$. Note also that $\mathcal{S}(\mathfrak{n})$ is a basis for $\mathfrak{a}^\vee$. (See A.3.)

We may identify elements of $\mathfrak{a}^\vee$ with elements of $\mathfrak{a}$ by means of the G-invariant scalar product on $\mathfrak{g}$, which is positive definite on $\mathfrak{a}$: for $Z, Z' \in \mathfrak{sl}_n(\mathbb{C})$,

$$\langle Z, Z' \rangle \;=\; \mathrm{Re}\,\mathrm{tr}(ZZ') \,.$$

In particular, for $\lambda \in \mathfrak{a}^\vee$, let H_λ be such that $\lambda(H) = \langle H_\lambda, H \rangle$ for all $H \in \mathfrak{a}$. Explicitly, for $\alpha_{i,j} \in \mathcal{R}(\mathfrak{n})$,

$$H_{i,j} \;=\; H_{\alpha_{i,j}} \;=\; E_{i,i} - E_{j,j} \;=\; \begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & 1 & & & & & \\ & & & & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & & \\ & & & & & & & -1 & \\ & & & & & & & & 0 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 0 \end{pmatrix} .$$

We define $\mathcal{R}(\mathfrak{n}_{G_P}) \subset \mathcal{R}(\mathfrak{n})$ such that:

$$\mathfrak{n}_{G_P} \;=\; \bigoplus_{\alpha \in \mathcal{R}(\mathfrak{n}_{G_P})} \mathfrak{n}_\alpha \;.$$

Explicitly, $\mathcal{R}(\mathfrak{n}_{G_P})$ consists of the characters $h_i - h_j$ where h_i and h_j lie in the *same block*. More precisely, using a partition $\{\mathscr{I}_k : 1 \le k \le r+1\}$ of the set $\{1, \ldots, n\}$ corresponding to the partition $\mathcal{P}$ of n, i.e. $\mathscr{I}_k = \{i : \; m_{k-1} + 1 \le i \le m_k\}$, we have $\mathcal{R}(\mathfrak{n}_{G_P}) = \{\alpha_{i,j} \in \mathcal{R}(\mathfrak{n}) |\; i, j \in \mathscr{I}_k \text{ for some } k\}$.

Alternately, we may define $\mathcal{R}(\mathfrak{n}_{G_P})$ as the subset of characters that vanish on $\mathfrak{a}_P$: $\{\alpha \in \mathcal{R}(\mathfrak{n}) : \; \alpha|_{\mathfrak{a}_P} \equiv 0\}$.

We define $\mathcal{S}(\mathfrak{n}_{G_P}) = \mathcal{R}(\mathfrak{n}_{G_P}) \cap \mathcal{S}(\mathfrak{n})$. Note that every element of $\mathcal{R}(\mathfrak{n}_{G_P})$ is a sum of elements of $\mathcal{S}(\mathfrak{n}_{G_P})$.

Next we define $\mathcal{R}(\mathfrak{n}_P) \subset \mathcal{R}(\mathfrak{n})$ such that

$$\mathfrak{n}_P \;=\; \bigoplus_{\alpha \in \mathcal{R}(\mathfrak{n}_P)} n_\alpha \;.$$

Alternately, $\mathcal{R}(\mathfrak{n}_P) = \mathcal{R}(\mathfrak{n}) \setminus \mathcal{R}(\mathfrak{n}_{G_P})$. (Here the backslash denotes a set difference, not a quotient.) Explicitly, $\mathcal{R}(\mathfrak{n}_P)$ consists of $\alpha_{i,j}$ where $i \in \mathscr{I}_k$ and $j \in \mathscr{I}_\ell$ for $k \neq \ell$, i.e. characters $h_i - h_j$ where h_i and h_j lie in *different blocks.*

Note that $\mathcal{R}(\mathfrak{n}_P)$ is *not* the same as the subset of characters in $\mathcal{R}(\mathfrak{n})$ whose restriction to $\mathfrak{a}_{GP}$ is zero.

We define $\mathcal{S}(\mathfrak{n}_P) = \mathcal{R}(\mathfrak{n}_P) \cap \mathcal{S}(\mathfrak{n})$. Then $\mathcal{S}(\mathfrak{n}_P) = \{\alpha_{m_k} \,:\, 1 \leq k \leq r\}$. Note that the elements of $\mathcal{R}(\mathfrak{n}_P)$ are in general *not* expressible as sums of elements of $\mathcal{S}(\mathfrak{n}_P)$.

At this point Jorgenson and Lang remark that $\mathcal{S}(\mathfrak{n}_P)$ is a basis of $\mathfrak{a}_P^\vee$. However this cannot literally be true, since $\mathcal{S}(\mathfrak{n}_P)$ is defined as a subset of $\mathcal{S}(\mathfrak{n})$, which is contained in $\mathfrak{a}^\vee$, but not $\mathfrak{a}_P^\vee$. There are two possible ways of interpreting this:

(1) Perhaps $\mathcal{S}(\mathfrak{n}_P)$ is a basis for the canonical isomorphic copy of $\mathfrak{a}_P^\vee$ inside $\mathfrak{a}^\vee$, namely $\widetilde{\mathfrak{a}_P^\vee} = \{\widetilde{\mu} = \mu \circ \pi_{\mathfrak{a}_P} \,:\, \mu \in \mathfrak{a}^\vee\}$, where $\pi_{\mathfrak{a}_P}$ is the projection $\mathfrak{a} \to \mathfrak{a}_P$.

(2) Perhaps restricting the elements of $\mathcal{S}(\mathfrak{n}_P)$ to $\mathfrak{a}_P$ yields a basis for $\mathfrak{a}_P^\vee$, i.e. perhaps the set $\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P} = \{\alpha|_{\mathfrak{a}_P} \,:\, \alpha \in \mathcal{S}(\mathfrak{n}_P)\}$ is a basis for $\mathfrak{a}_P^\vee$.

We can easily see that (1) fails by looking at an example. Let $G = SL_5(\mathbb{C})$, $P = P^{3,2}$. Consider $\alpha_3 \in \mathcal{S}(\mathfrak{n}_P)$ and $H \in \mathfrak{a}_{G_P}$. Then

$$\alpha_3(H) \;=\; \alpha_3\left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right) \;=\; h_3 - h_4 \;.$$

The fact that $H \in \mathfrak{a}_{G_P}$ implies $h_1 + h_2 + h_3 = 0$ and $h_4 + h_5 = 0$. But these relations do not force $\alpha_3(H) = 0$. Consider for example, $H = H_1 + H_4 \in \mathfrak{a}_{G_P}$,

$$\alpha_3(H) \;=\; \alpha_3\left(\begin{array}{ccc|cc} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ \hline & & & 1 & \\ & & & & -1 \end{array}\right) \;=\; -1 \;\neq\; 0 \;.$$

Then $\alpha_3 \in \mathcal{S}(\mathfrak{n}_P)$, but $\alpha_3 \notin \widetilde{\mathfrak{a}_P^\vee}$, since $\alpha_3|_{\mathfrak{a}_{G_P}} \not\equiv 0$.

As it turns out, (2) is true; see A.5. Note that, along with the isomorphism $\mathfrak{a}_P^\vee \cong \widetilde{\mathfrak{a}_P^\vee}$, this implies that $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}} = \{\widetilde{\alpha|_{\mathfrak{a}_P}} = \alpha|_{\mathfrak{a}_P} \circ \pi_{\mathfrak{a}_P} \,:\, \alpha \in \mathcal{S}(\mathfrak{a}_P)\}$ is a basis for $\widetilde{\mathfrak{a}_P^\vee}$.

Jorgenson and Lang also state that the elements of $\mathcal{S}(\mathfrak{n}_P)$ can be indexed in the form $\alpha_{P,1}, \ldots, \alpha_{P,r}$. However, we choose to notate the elements of $\mathcal{S}(\mathfrak{n}_P)$ as $\alpha_{m_1}, \ldots, \alpha_{m_r}$ to be clear that we are referring to elements of $\mathcal{S}(\mathfrak{n}_P) \subset \mathcal{S}(\mathfrak{n}) = \{\alpha_i \,:\, 1 \leq i \leq n-1\}$. Instead, we prefer to use the notation $\alpha_{P,1}, \ldots, \alpha_{P,r}$ to refer to elements of the basis $\widetilde{\mathcal{S}(\mathfrak{n}_P)}|_{\mathfrak{a}_P}$ of $\widetilde{\mathfrak{a}_P^\vee}$. More precisely, we let $\alpha_{P,k} = \widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}}$. As discussed in A.6, $\alpha_{P,k}$ may be explicitly described as:

$$\alpha_{P,k} \;=\; \left(\begin{array}{c}\text{avg. of entries}\\ \text{in } k\text{th block}\end{array}\right) - \left(\begin{array}{c}\text{avg. of entries in}\\ (k+1)\text{th block}\end{array}\right) .$$

3 Orthogonality

3.1 Orthogonality of certain characters

After defining the relevant characters and simple characters, Jorgenson and Lang point out that

$$\mathcal{R}(\mathfrak{n}) \;=\; \mathcal{R}(\mathfrak{n}_{G_P}) \sqcup \mathcal{R}(\mathfrak{n}_P)\ .$$

This disjoint union is clear from the explicit descriptions of these sets. Referencing this equation, which they label (3), Jorgenson and Lang state, "Directly from the definitions, we see that the decomposition (3) is an orthogonal decomposition." The intended meaning of this statement is unclear.

One interpretation of the statement would be that all the characters in $\mathcal{R}(\mathfrak{n}_{G_P})$ are orthogonal to all the characters in $\mathcal{R}(\mathfrak{n}_P)$. However, this is clearly false, as can easily been seen by looking at an example. With $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$, we have $\alpha_{2,3} \in \mathcal{R}(\mathfrak{n}_{G_P})$ and $\alpha_{3,4} \in \mathcal{R}(\mathfrak{n}_P)$, but

$$\langle \alpha_{2,3}, \alpha_{3,4}\rangle \;=\; \alpha_{2,3}(H_{3,4}) \;=\; \alpha_{2,3}\begin{pmatrix} 0 &&&& \\ & 0 &&& \\ && 1 && \\ &&& -1 & \\ &&&& 0 \end{pmatrix} \;=\; -1 \;\neq\; 0\ .$$

On the other hand, it seems odd to refer to a disjoint union of sets as a "decomposition," so perhaps the reference to Equation 3 is a misprint.

Perhaps they intended to refer to the orthogonal decomposition $\mathfrak{a}^\vee = \widetilde{\mathfrak{a}_P^\vee} \oplus \widetilde{\mathfrak{a}_{G_P}^\vee}$. This seems unlikely though since, as shown above, $\mathcal{R}(\mathfrak{n}_P) \nsubseteq \widetilde{\mathfrak{a}_P^\vee}$ (although $\mathcal{R}(\mathfrak{n}_{G_P}) \subset \widetilde{\mathfrak{a}_{G_P}^\vee}$).

Perhaps Jorgenson and Lang are referring to the orthogonal decomposition $\mathfrak{n} = \mathfrak{n}_P \oplus \mathfrak{n}_{G_P}$. This does follow quickly from the definitions, but it also seems rather trivial and unrelated to the discussion that follows.

See A.8 for a list of true statements one can make regarding orthogonality of characters in $\mathfrak{a}^\vee$.

3.2 Orthogonal decomposition $\rho = \rho_{G_P} + \rho_P$

As usual $\rho \in \mathfrak{a}^\vee$ is defined to be the half-sum of positive roots, although Jorgenson and Lang use a different terminology, referring to it as the half-trace of the regular representation of $\mathfrak{a}$ on $\mathfrak{n}$. For $SL_n(\mathbb{C})$, since the multiplicity of each root is two, we have

$$\rho \;=\; \rho_G \;=\; \tfrac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathfrak{n})} m(\alpha)\alpha \;=\; \sum_{\alpha \in \mathcal{R}(\mathfrak{n})} \alpha\ .$$

Analogously, we define

$$\rho_{G_P} \;=\; \sum_{\alpha \in \mathcal{R}(\mathfrak{n}_{G_P})} \alpha \quad \text{and} \quad \rho_P \;=\; \sum_{\alpha \in \mathcal{R}(\mathfrak{n}_P)} \alpha\ .$$

Then clearly $\rho_G = \rho_{G_P} + \rho_P$, since $\mathcal{R}(\mathfrak{n}) = \mathcal{R}(\mathfrak{n}_{G_P}) \sqcup \mathcal{R}(\mathfrak{n}_P)$.

Note that Jorgenson and Lang reference an equation labeled (4) which states the disjoint union $\mathcal{S}(\mathfrak{n}) = \mathcal{S}(\mathfrak{n}_{G_P}) \sqcup \mathcal{S}(\mathfrak{n}_P)$, to justify the decomposition $\rho_G = \rho_{G_P} + \rho_P$. However, the decomposition follows rather from (3), which states the corresponding disjoint union for relevant characters. This is another reason to question the reference to (3) in the unclear comment, discussed above, about the "orthogonal decomposition."

Further, (and this is Jorgenson and Lang's Lemma 5.1), ρ_P is orthogonal to all $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$ and thus to ρ_{G_P}, since ρ_{G_P} is a sum of relevant characters in $\mathcal{R}(\mathfrak{n}_{G_P})$ and every relevant character in $\mathcal{R}(\mathfrak{n}_{G_P})$ is a sum of simple characters in $\mathcal{S}(\mathfrak{n}_{G_P})$.

We briefly outline the proof; the appendices fill in the necessary details. See in particular B.2.

Using the "dual basis" $\mathcal{S}(\mathfrak{n})'$ of $\mathfrak{a}^\vee$, one can show that $\langle \rho_G, \alpha\rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n})$; see A.3 and B.2. Since $\mathfrak{g}_{G_P} \cong \mathfrak{sl}_{n_1}(\mathbb{C}) \oplus \dots \mathfrak{sl}_{n_{r+1}}(\mathbb{C})$, a similar argument can be used to show that $\langle \rho_{G_P}, \alpha\rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$; see B.2. Then, for all $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$,

$$\langle \rho_P, \alpha\rangle \;=\; \langle \rho_G - \rho_{G_P}, \alpha\rangle \;=\; \langle \rho_G, \alpha\rangle \;-\; \langle \rho_{G_P}, \alpha\rangle \;=\; 2-2 \;=\; 0\,.$$

Note that there appears to be a misprint in Jorgenson and Lang's equation (7P). It states $\langle \rho_G, \alpha\rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$. This clearly true, following trivially from the fact that $\langle \rho_G, \alpha\rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n})$. However, the stronger statement, that $\langle \rho_{G_P}, \alpha\rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$, seems to be needed for the proof.

Appendices

A Subspaces and Bases for $\mathfrak{a}$ and $\mathfrak{a}^\vee$

In this appendix, we discuss the orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_P \oplus \mathfrak{a}_{G_P}$ and the corresponding orthogonal decomposition of $\mathfrak{a}^\vee$, providing explicit descriptions of the projection maps and of bases and dual bases for the various subspaces.

A.1 Orthogonal decomposition of $\mathfrak{a}$

The Lie subalgebra $\mathfrak{a}$ decomposes as a direct sum: $\mathfrak{a} = \mathfrak{a}_{G_P} \oplus \mathfrak{a}_P$. Given any $H \in \mathfrak{a}$, we may write $H = H_{\mathfrak{a}_P} + H_{\mathfrak{a}_{G_P}}$, as follows.

Let $h_1, \dots, h_n$ be the (diagonal) entries of H. Then the jth entry of $H_{\mathfrak{a}_P}$, lying in the kth block, (i.e. $m_{k-1} + 1 \le j \le m_k$ for some k in the range $1 \le k \le r+1$), is given by:

$$h'_j \;=\; \frac{1}{n_k} \sum_{\ell=m_{k-1}+1}^{m_k} h_\ell\,,$$

i.e. each of the entries in the kth block of $H_{\mathfrak{a}_P}$ is equal to the average of the entries in the kth block of H. The jth (diagaonal) entry of $H_{\mathfrak{a}_{G_P}}$, lying in the kth block, is

$$h''_j \;=\; h_j \;-\; \frac{1}{n_k} \sum_{\ell=m_{k-1}+1}^{m_k} h_\ell \;=\; \frac{1}{n_k} \sum_{\ell=m_{k-1}+1}^{m_k} (h_j - h_\ell) \;=\; \frac{(n_k-1)h_j}{n_k} \;-\; \sum_{\ell \ne j} h_\ell\,,$$

where the last summation runs over all $\ell \ne j$ in the range $m_{k-1} + 1 \;\le\; \ell \;\le\; m_k$.

For example, for $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$, we have:

$$\left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right) \;=\; \left(\begin{array}{ccc|cc} \frac{h_1+h_2+h_3}{3} & & & & \\ & \frac{h_1+h_2+h_3}{3} & & & \\ & & \frac{h_1+h_2+h_3}{3} & & \\ \hline & & & \frac{h_4+h_5}{2} & \\ & & & & \frac{h_4+h_5}{2} \end{array}\right)$$

$$+ \left(\begin{array}{ccc|cc} \frac{2h_1-h_2-h_3}{3} & & & & \\ & \frac{-h_1+2h_2-h_3}{3} & & & \\ & & \frac{-h_1-h_2+2h_3}{3} & & \\ \hline & & & \frac{h4-h_5}{2} & \\ & & & & \frac{-h_4+h_5}{2} \end{array}\right).$$

A.2 Orthogonal decomposition of $\mathfrak{a}^\vee$.

The orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_P \oplus \mathfrak{a}_{G_P}$ gives rise to an orthogonal decomposition of the dual space $\mathfrak{a}^\vee$ into subspaces isomorphic to $\mathfrak{a}_P^\vee$ and $\mathfrak{a}_{G_P}^\vee$. In particular, we have $\mathfrak{a}^\vee = \widetilde{\mathfrak{a}_P^\vee} \oplus \widetilde{\mathfrak{a}_{G_P}^\vee}$ with

$$\mathfrak{a}_P^\vee \;\cong\; \widetilde{\mathfrak{a}_P^\vee} \;=\; \{\tilde{\mu} = \mu \circ \pi_{\mathfrak{a}_P} : \mu \in \mathfrak{a}_P^\vee\} \;=\; \{\lambda \in \mathfrak{a}^\vee \,:\, \lambda_{\mathfrak{a}_{G_P}} \equiv 0\} \;\subset\; \mathfrak{a}^\vee \,,$$

$$\mathfrak{a}_{G_P}^\vee \;\cong\; \widetilde{\mathfrak{a}_{G_P}^\vee} \;=\; \{\tilde{\mu} = \mu \circ \pi_{\mathfrak{a}_{G_P}} : \mu \in \mathfrak{a}_{G_P}^\vee\} \;=\; \{\lambda \in \mathfrak{a}^\vee \,:\, \lambda_{\mathfrak{a}_P} \equiv 0\} \;\subset\; \mathfrak{a}^\vee \,,$$

where $\pi_{\mathfrak{a}_P}$ and $\pi_{\mathfrak{a}_{G_P}}$ denote the projections from $\mathfrak{a}$ to $\mathfrak{a}_P$ and $\mathfrak{a}_{G_P}$, respectively.

A.3 Bases for $\mathfrak{a}$ and $\mathfrak{a}^\vee$

One basis for $\mathfrak{a}$ is $\mathscr{B} = \{H_i \mid 1 \le i \le n-1\}$, where

$$H_i \;=\; H_{\alpha_i} \;=\; E_{i,i} - E_{i+1,i+1} \;=\; \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix} .$$

The notation H_i is standard, but the usage of $\mathscr{B}$ to denote the collection of such H_i is not. It will be useful to be able to refer to $\mathscr{B}$ when considering bases of subspaces of $\mathfrak{a}$ of their dual spaces.

This gives rise to a basis for $\mathfrak{a}^\vee$ under the identification of $\mathfrak{a}$ with $\mathfrak{a}^\vee$ via $\langle\, ,\, \rangle$. The corresponding basis for $\mathfrak{a}^\vee$ is $\mathcal{S}(\mathfrak{n}) = \{\alpha_i \,:\, 1 \le i \le n-1\}$, where, as above, $\alpha_i(H) = h_i - h_{i+1}$.

The basis $\mathscr{B} = \{H_i\}$ for $\mathfrak{a}$ also gives rise to a dual basis $\mathcal{S}(\mathfrak{n})' = \{\alpha_i'\}$ for $\mathfrak{a}^\vee$. (See **??**.)

$$\alpha_i'(H) \;=\; h_1 + \cdots + h_i \,.$$

Finally, we may identify each α_i' with an element H_i' of $\mathfrak{a}$ via $\langle\, ,\, \rangle$, to obtain another basis $\mathscr{B}'$ of $\mathfrak{a}$, which is also "dual" to $\mathscr{B}$ in the sense that $\langle H_i, H_j' \rangle = \delta_{i,j}$.

$$H_i' \;=\; \begin{pmatrix} 1-\frac{i}{n} & & & & & \\ & \ddots & & & & \\ & & 1-\frac{i}{n} & & & \\ & & & -\frac{i}{n} & & \\ & & & & \ddots & \\ & & & & & -\frac{i}{n} \end{pmatrix} ,$$

where the transition from $1 - \frac{i}{n}$ to $-\frac{i}{n}$ happens from the ith to the $(i+1)$th diagonal entry.

For example, with $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$,

$$H_1' = \begin{pmatrix} \frac45 & & & & \\ & -\frac15 & & & \\ & & -\frac15 & & \\ & & & -\frac15 & \\ & & & & -\frac15 \end{pmatrix}, \quad H_2' = \begin{pmatrix} \frac35 & & & & \\ & \frac35 & & & \\ & & -\frac25 & & \\ & & & -\frac25 & \\ & & & & -\frac25 \end{pmatrix}, \quad H_3' = \begin{pmatrix} \frac25 & & & & \\ & \frac25 & & & \\ & & \frac25 & & \\ & & & -\frac35 & \\ & & & & -\frac35 \end{pmatrix}, \quad H_4' = \begin{pmatrix} \frac15 & & & & \\ & \frac15 & & & \\ & & \frac15 & & \\ & & & \frac15 & \\ & & & & -\frac45 \end{pmatrix}.$$

We have explicit descriptions of the bases $\{\alpha_i\}$ and $\{\alpha_i'\}$ for $\mathfrak{a}^\vee$ as functions of $h_1, \ldots, h_n$, but we may also want to express α_i as a linear combination of elements of $\{\alpha_i'\}$ or vice versa.

$$\alpha_i' \;=\; \left(\tfrac{n-i}{n}\right)\sum_{j=1}^{i} j\,\alpha_j \;+\; \left(\tfrac{i}{n}\right)\sum_{j=i+1}^{n-1}(n-j)\,\alpha_j\ .$$

For example, for $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$,

$$\begin{aligned}
\alpha_1' &= \tfrac{4}{5}\big(\alpha_1\big) + \tfrac{1}{5}\big(3\alpha_2 + 2\alpha_3 + \alpha_4\big)\\
\alpha_2' &= \tfrac{3}{5}\big(\alpha_1 + 2\alpha_2\big) + \tfrac{2}{5}\big(2\alpha_3 + \alpha_4\big)\\
\alpha_3' &= \tfrac{2}{5}\big(\alpha_1 + 2\alpha_2 + 3\alpha_3\big) + \tfrac{3}{5}\big(\alpha_4\big)\\
\alpha_4' &= \tfrac{1}{5}\big(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4\big)\ .
\end{aligned}$$

Conversely, we may write α_i as a linear combination of elements of $\mathcal{S}(\mathfrak{n})'$, as follows:

$$\alpha_i \;=\; -\alpha_{i-1}' + 2\alpha_i' - \alpha_{i+1}',$$

where $\alpha_0' = \alpha_n' = 0$.

For example, for $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$,

$$\alpha_1 = 2\alpha_1' - \alpha_2'\,, \quad \alpha_2 = -\alpha_1' + 2\alpha_2' - \alpha_3'\,, \quad \alpha_3 = -\alpha_2' + 2\alpha_3' - \alpha_4'\,, \quad \alpha_4 = -\alpha_3' + 2\alpha_4'\ .$$

A.4 Bases for $\mathfrak{a}_P$ and $\mathfrak{a}_{G_P}$

We have two bases $\mathscr{B} = \{H_i\}$ and $\mathscr{B}' = \{H_i'\}$ for $\mathfrak{a}$, and we wish to find bases for the orthogonal subspaces $\mathfrak{a}_{G_P}$ and $\mathfrak{a}_P$.

Note that for $1 \le i \le n-1$, $i \ne m_k$, for any $1 \le k \le r$, $H_i \in \mathfrak{a}_{G_P}$. In fact such H_i span $\mathfrak{a}_{G_P}$, so form a basis for $\mathfrak{a}_{G_P}$.

For example, for $G = SL_5(\mathbb{C})$, $P = P^{3,2}$,

$$H_1 = \left(\begin{array}{ccc|cc} 1 &&&& \\ & -1 &&& \\ && 0 && \\ \hline &&& 0 & \\ &&&& 0 \end{array}\right), \quad H_2 = \left(\begin{array}{ccc|cc} 0 &&&& \\ & 1 &&& \\ && -1 && \\ \hline &&& 0 & \\ &&&& 0 \end{array}\right), \quad H_4 = \left(\begin{array}{ccc|cc} 0 &&&& \\ & 0 &&& \\ && 0 && \\ \hline &&& 1 & \\ &&&& -1 \end{array}\right)$$

form a basis for $\mathfrak{a}_{G_P}$.

Thus we see that intersecting the basis $\mathscr{B}$ for $\mathfrak{a}$ with the subspace $\mathfrak{a}_{G_P}$ yields a basis for $\mathfrak{a}_{G_P}$. One might naively hope that the complement $\{H_i \,:\, i = m_k,\ 1 \le k \le r\}$ yields a basis for $\mathfrak{a}_P$, but this is not so since such H_i do not even lie in $\mathfrak{a}_P$. However, we may obtain a basis for $\mathfrak{a}_P$ by projecting such H_i to $\mathfrak{a}_P$.

Using the method described in A.1, we find the projection $\pi_{\mathfrak{a}_P}(H_i) = \big(H_{m_k}\big)_{\mathfrak{a}_P}$ of H_i with $i = m_k$ to $\mathfrak{a}_P$. The matrix $\big(H_{m_k}\big)_{\mathfrak{a}_P}$ is block diagonal, with all blocks on the diagonal being zero except the kth block which is $\frac{1}{n_k}\mathbb{1}_{n_k}$ and the $(k+1)$th block which is $-\frac{1}{n_{k+1}}\mathbb{1}_{n_{k+1}}$.

In our example, $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, we have

$$\big(H_3\big)_{\mathfrak{a}_P} = \left(\begin{array}{ccc|cc} \frac{1}{3} &&&& \\ & \frac{1}{3} &&& \\ && \frac{1}{3} && \\ \hline &&& -\frac{1}{2} & \\ &&&& -\frac{1}{2} \end{array}\right).$$

To see the structure more generally, let us consider another example, say $G = SL_{12}(\mathbb{C})$, $P = P^{4,3,3,2}$. Then the following three matrices form a basis for $\mathfrak{a}_P$.

$$(H_4)_{\mathfrak{a}_P} = \begin{pmatrix} \boxed{\frac{1}{4}\mathbb{1}_4} & & & 0 \\ & \boxed{-\frac{1}{3}\mathbb{1}_3} & & \\ & & \boxed{\mathbb{0}_3} & \\ 0 & & & \boxed{\mathbb{0}_2} \end{pmatrix}, \quad (H_7)_{\mathfrak{a}_P} = \begin{pmatrix} \boxed{\mathbb{0}_4} & & & 0 \\ & \boxed{\frac{1}{3}\mathbb{1}_3} & & \\ & & \boxed{-\frac{1}{3}\mathbb{1}_3} & \\ 0 & & & \boxed{\mathbb{0}_2} \end{pmatrix}, \quad (H_{10})_{\mathfrak{a}_P} = \begin{pmatrix} \boxed{\mathbb{0}_4} & & & 0 \\ & \boxed{\mathbb{0}_3} & & \\ & & \boxed{\frac{1}{3}\mathbb{1}_3} & \\ 0 & & & \boxed{-\frac{1}{2}\mathbb{1}_2} \end{pmatrix}.$$

Now we look for bases of $\mathfrak{a}_{G_P}$ and $\mathfrak{a}_P$ related to the basis $\mathscr{B}' = \{H'_i\}$ of $\mathfrak{a}$. Note that for $i = m_k$ for some $1 \le k \le r$, $H'_i \in \mathfrak{a}_P$. Since $\mathscr{B}' \cap \mathfrak{a}_P = \{H'_i \,:\, i = m_k,\, 1 \le k \le r\}$ is linearly independent (being a subset of the basis $\mathscr{B}'$) and has cardinality $r = \dim_{\mathbb{R}}(\mathfrak{a}_P)$, it is a basis for $\mathfrak{a}_P$.

In our small example, $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, a basis for $\mathfrak{a}_P$ is given by the intersection $\mathscr{B}' \cap \mathfrak{a}_P = \{H'_3\}$, where

$$H'_3 = \left(\begin{array}{ccc|cc} \frac{2}{5} & & & & \\ & \frac{2}{5} & & & \\ & & \frac{2}{5} & & \\ \hline & & & -\frac{3}{5} & \\ & & & & -\frac{3}{5} \end{array}\right).$$

Now we aim to find a basis for $\mathfrak{a}_{G_P}$ from $\mathscr{B}' = \{H'_i\}$. The intersection of $\mathscr{B}'$ with $\mathfrak{a}_{G_P}$ is empty, but we may construct a basis for $\mathfrak{a}_{G_P}$ by projecting H'_i $(i \ne m_k, 1 \le k \le r)$ to $\mathfrak{a}_{G_P}$.

For $i = m_{k-1} + \ell$ with $1 \le \ell \le n_k - 1$ (so $m_{k-1} < i < m_k$), the projection of H'_i to $\mathfrak{a}_{G_P}$ is zero except for in the kth block, which is:

$$\big(k\text{th block of } (H'_i)_{\mathfrak{a}_{G_P}}\big) = \begin{pmatrix} 1-\ell/n_k & & & & & \\ & \ddots & & & & \\ & & 1-\ell/n_k & & & \\ & & & -\ell/n_k & & \\ & & & & \ddots & \\ & & & & & -\ell/n_k \end{pmatrix},$$

where the transition from $1 - \ell/n_k$ to $-\ell/n_k$ occurs at the transition from the ith to the $(i+1)$st position.

Recall that $\mathfrak{a}_{G_P}$ decomposes into a direct sum of (copies of) subalgebras of $\mathfrak{sl}_{n_1}(\mathbb{C}), \ldots, \mathfrak{sl}_{n_{r+1}}(\mathbb{C})$, where for $1 \le k \le r+1$, the subalgrebra of $\mathfrak{sl}_{n_k}(\mathbb{C})$ that (whose copy) occurs in the direct sum is $\mathfrak{a}_{\mathfrak{sl}_{n_k}(\mathbb{C})}$, the subalgebra of diagonal $n_k \times n_k$ matrices of trace zero. Clearly $\{(H'_i)_{\mathfrak{a}_{G_P}} \,:\, i = m_{k-1} + \ell,\, 1 \le \ell \le n_k - 1\}$ is a basis for the copy of $\mathfrak{a}_{\mathfrak{sl}_{n_k}(\mathbb{C})}$ inside $\mathfrak{a}_{G_P}$, so $\{(H'_i)_{\mathfrak{a}_{G_P}} \,:\, 1 \le i \le n-1,\, i \ne m_k\}$ is a basis for $\mathfrak{a}_{G_P}$.

Returning to our example $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, a basis for $\mathfrak{a}_{G_P}$ consists of:

$$(H'_1)_{\mathfrak{a}_{G_P}} = \left(\begin{array}{ccc|cc} \frac{2}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ \hline & & & 0 & \\ & & & & 0 \end{array}\right), \quad (H'_2)_{\mathfrak{a}_{G_P}} = \left(\begin{array}{ccc|cc} \frac{1}{3} & & & & \\ & \frac{1}{3} & & & \\ & & -\frac{2}{3} & & \\ \hline & & & 0 & \\ & & & & 0 \end{array}\right), \quad (H'_4)_{\mathfrak{a}_{G_P}} = \left(\begin{array}{ccc|cc} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ \hline & & & \frac{1}{2} & \\ & & & & -\frac{1}{2} \end{array}\right).$$

A.5 Bases for $\mathfrak{a}_{G_P}^\vee$ and $\mathfrak{a}_P^\vee$

A.5.1 Bases for $\mathfrak{a}_{G_P}^\vee$

First we look for a basis for $\mathfrak{a}_{G_P}^\vee$ related to $\mathcal{S}(\mathfrak{n}_{G_P})$. Since elements of $\mathcal{S}(\mathfrak{n}_{G_P})$ are linear maps $\mathfrak{a} \to \mathbb{R}$, rather than $\mathfrak{a}_{G_P} \to \mathbb{R}$, we know that $\mathcal{S}(\mathfrak{n}_{G_P})$ itself cannot be a basis for $\mathfrak{a}_{G_P}$. However, restricting elements of $\mathcal{S}(\mathfrak{n}_{G_P})$ to $\mathfrak{a}_{G_P}$ gives a basis for $\mathfrak{a}_{G_P}^\vee$, as is clear from the fact that

$$\mathscr{B} \cap \mathfrak{a}_{G_P} \;=\; \{H_i \,:\, 1 \le i \le n-1,\, i \ne m_k \text{ for any } 1 \le k \le r\}$$

is a basis for $\mathfrak{a}_{G_P}$. We denote the set of such restrictions by $\mathcal{S}(\mathfrak{n}_{G_P})|_{\mathfrak{a}_{G_P}}$.

For example, when $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$, a basis for $\mathfrak{a}_{G_P}^\vee$ is given by:

$$\alpha_1|_{\mathfrak{a}_{G_P}}(H) = h_1 - h_2, \quad \alpha_2|_{\mathfrak{a}_{G_P}}(H) = h_2 - h_3, \quad \alpha_3|_{\mathfrak{a}_{G_P}}(H) = h_4 - h_5\,,$$

where $H = \left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right)$ with $h_1 + h_2 + h_3 = 0$ and $h_4 + h_5 = 0$.

A basis for $\mathfrak{a}_{G_P}^\vee$ dual to $\mathcal{S}(\mathfrak{a}_{G_P})|_{\mathfrak{a}_{G_P}}$ is obtained by restricting the elements of $\mathcal{S}(\mathfrak{n}_{G_P})'$, defined as $\{\alpha' \in \mathcal{S}(\mathfrak{n})' \,:\, \alpha \in \mathcal{S}(\mathfrak{n}_{G_P})\}$, to $\mathfrak{a}_{G_P}$. We denote this basis by $\mathcal{S}(\mathfrak{n}_{G_P})'|_{\mathfrak{a}_{G_P}}$.

We describe these characters explicitly as functions of $H \in \mathfrak{a}_{G_P}$. For $\alpha_i \in \mathcal{S}(\mathfrak{n}_{G_P})$, we have i and $i+1$ within the same partition $\mathscr{I}_k = \{m_{k-1} < i \le m_k\}$ of $\{1, \dots, n\}$, i.e. there is some k in $1 \le k \le r+1$, such that $m_{k-1} < i < m_k$. Then for $H \in \mathfrak{a}_{G_P}$, with (diagonal) entries $h_1, \dots, h_n$,

$$\alpha_i'(H) \;=\; \alpha_i' \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} \;=\; h_1 + \dots + h_{m_{k-1}} + h_{m_{k-1}+1} + \dots + h_i \;=\; h_{m_{k-1}+1} + \dots + h_i\,,$$

since the trace of each block in H is zero. In other words, for i in the range $m_{k-1} < i < m_k$, the character $\alpha_i'|_{\mathfrak{a}_{G_P}}$ takes the sum of the (diagonal) entries in the kth block of a matrix $H \in \mathfrak{a}_{G_P}$ up to and including the ith (diagonal) entry.

For example, for $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, let $H = \left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right) \in \mathfrak{a}_{G_P}$, i.e. $h_1 + h_2 + h_3 = 0$ and $h_4 + h_5 = 0$. Then the basis $\mathcal{S}(\mathfrak{n}_{G_P})'|_{\mathfrak{a}_{G_P}}$ for $\mathfrak{a}_{G_P}^\vee$ consists of:

$$\alpha_1'|_{\mathfrak{a}_{G_P}}(H) \;=\; h_1\,, \quad \alpha_2'|_{\mathfrak{a}_{G_P}}(H) \;=\; h_1 + h_2\,, \quad \alpha_4'|_{\mathfrak{a}_{G_P}}(H) \;=\; h_4\,.$$

A.5.2 Bases for $\mathfrak{a}_P^\vee$

Next we describe bases for $\mathfrak{a}_P^\vee$ corresponding to $\mathcal{S}(\mathfrak{n}_P)$ and $\mathcal{S}(\mathfrak{n}_P)'$ respectively.

A basis for $\mathfrak{a}_P$ is obtained by restricting elements of $\mathcal{S}(\mathfrak{n}_P)$ to $\mathfrak{a}_P$; we denote the set of such restrictions by $\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}$. A dual basis is obtained by restricting elements of $\mathcal{S}(\mathfrak{n}_P)' = \{\alpha' \,:\, \alpha \in \mathcal{S}(\mathfrak{n}_P)\}$ to $\mathfrak{a}_P$.

Explicitly, for $i = m_k$, with $1 \le k \le r$, for $H \in \mathfrak{a}_P$, with diagonal entries constant on each block, with h_ℓ being the constant entry of the ℓth block,

$$\alpha_i'(H) \;=\; n_1 h_1 \,+\, n_2 h_2 \,+\, \dots \,+\, n_k h_k \qquad \text{(where } h_\ell \text{ is the constant diagonal entry of the } \ell\text{th block.)}$$

We may prefer to index the entries of $H \in \mathfrak{a}_P$ so that the jth entry is h_j. In this case, the fact that the entries are constant on each block implies $h_1 = h_2 = \dots = h_{m_1}$, $h_{m_1+1} = h_{m_1+2} = \dots = h_{m_2}$, ..., $h_{m_r+1} = h_{m_r+2} = \dots = h_{m_{r+1}} = h_n$, and the formula may be written

$$\alpha_i'(H) \;=\; n_1 h_1 \,+\, n_2 h_{m_1+1} \,+\, \dots \,+\, n_k h_{m_{k-1}+1} \qquad \text{(where } h_j \text{ is the } j\text{th diagonal entry of } H.)$$

For example, for $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$, and for $H \in \mathfrak{a}_P$, say $H = \left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right)$ with $3h_1 + 2h_4 = 0$,

the basis $\mathcal{S}(\mathfrak{n}_P)'_{\mathfrak{a}_P}$ for $\mathfrak{a}_P^\vee$ consists of the single character:

$$\alpha_3'|_{\mathfrak{a}_P} \left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_1 & & & \\ & & h_1 & & \\ \hline & & & h_4 & \\ & & & & h_4 \end{array}\right) \;=\; 3h_1\ .$$

To see the structure more generally, let us consider another example, say $G = SL_{12}(\mathbb{C})$, $P = P^{4,3,3,2}$. Then $H \in \mathfrak{a}_P$ may be written in the form

$$H \;=\; \begin{pmatrix} h_1\mathbb{1}_4 & & & 0 \\ & h_5\mathbb{1}_3 & & \\ & & h_8\mathbb{1}_3 & \\ 0 & & & h_{11}\mathbb{1}_2 \end{pmatrix}, \qquad \text{where} \quad 4h_1 + 3h_5 + 3h_8 + 2h_{11} \;=\; 0\ .$$

In this case, $\mathcal{S}(\mathfrak{n}_P) = \{\alpha_4, \alpha_7, \alpha_{10}\}$ and the basis $\mathcal{S}(\mathfrak{n}_P)'_{\mathfrak{a}_P}$ for $\mathfrak{a}_P^\vee$ consists of

$$\alpha_4'|_{\mathfrak{a}_P}(H) \;=\; 4h_1\ , \quad \alpha_7'|_{\mathfrak{a}_P}(H) \;=\; 4h_1 + 3h_5\ , \quad \alpha_{10}'|_{\mathfrak{a}_P}(H) \;=\; 4h_1 + 3h_5 + 3h_8\ .$$

A.6 Bases for $\widetilde{\mathfrak{a}_{G_P}^\vee}$ and $\widetilde{\mathfrak{a}_P^\vee}$

Now we consider bases for the isomorphic copies $\widetilde{\mathfrak{a}_{G_P}^\vee}$ and $\widetilde{\mathfrak{a}_P^\vee}$ of $\mathfrak{a}_{G_P}^\vee$ and $\mathfrak{a}_P^\vee$, respectively, inside $\mathfrak{a}^\vee$.

First we introduce some notation. Recall that any $H \in \mathfrak{a}$ may be written uniquely in the form $H = H_{\mathfrak{a}_P} + H_{\mathfrak{a}_{G_P}}$, where $H_{\mathfrak{a}_P} = \pi_{\mathfrak{a}_P}(H) \in \mathfrak{a}_P$ and likewise $H_{\mathfrak{a}_{G_P}}$ is the projection of H to $\mathfrak{a}_{\mathfrak{a}_{G_P}}$. (See A.1.) For $\mu \in \mathfrak{a}_{G_P}^\vee$, we extend μ to a function $\widetilde{\mu}$ on $\mathfrak{a}$ by

$$\widetilde{\mu}(H) \;=\; \widetilde{\mu}(H_{\mathfrak{a}_P} + H_{\mathfrak{a}_{G_P}}) \;=\; \mu(H_{\mathfrak{a}_{G_P}})\ ,$$

so in particular, $\widetilde{\mu}|_{\mathfrak{a}_P} \equiv 0$. Similarly, we may extend elements of $\mathfrak{a}_P^\vee$ to linear functions on $\mathfrak{a}$ whose restrictions to $\mathfrak{a}_{G_P}$ are zero. We use the tilde notation to denote these extensions also.

A.6.1 Bases for $\widetilde{\mathfrak{a}_{G_P}^\vee}$

Since $\widetilde{\mathfrak{a}_{G_P}^\vee} \cong \mathfrak{a}_{G_P}^\vee$, and as we have shown above in A.5, $\mathcal{S}(\mathfrak{n}_{G_P})|_{\mathfrak{a}_{G_P}}$ is a basis for $\mathfrak{a}_{G_P}^\vee$, it is clear that

$$\mathcal{S}(\widetilde{\mathfrak{n}_{G_P})|_{\mathfrak{a}_{G_P}}} \;=\; \{\widetilde{\alpha|_{\mathfrak{a}_{G_P}}}\ :\ \alpha \in \mathcal{S}(\mathfrak{n}_{G_P})\}$$

is a basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$. However, we also know that $\mathcal{S}(\mathfrak{n}_{G_P}) = \{\alpha \in \mathcal{S}(\mathfrak{n})\ :\ \alpha|_{\mathfrak{a}_P} \equiv 0\}$, so in fact $\mathcal{S}(\mathfrak{n}_{G_P}) = \mathcal{S}(\widetilde{\mathfrak{n}_{G_P})|_{\mathfrak{a}_{G_P}}}$, and it is simpler to say that $\mathcal{S}(\mathfrak{n}_{G_P})$ is a basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$.

Similarly, since $\mathcal{S}(\mathfrak{n}_{G_P})'|_{\mathfrak{a}_{G_P}}$ is a basis for $\mathfrak{a}_{G_P}^\vee$ dual to $\mathcal{S}(\mathfrak{n}_{G_P})|_{\mathfrak{a}_{G_P}}$ as shown in A.5, we can see that $\mathcal{S}(\widetilde{\mathfrak{n}_{G_P})'|_{\mathfrak{a}_{G_P}}}$ is a basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$ that is dual to $\mathcal{S}(\mathfrak{n}_{G_P}) = \mathcal{S}(\widetilde{\mathfrak{n}_{G_P})|_{\mathfrak{a}_{G_P}}}$. Note, however, that characters in $\mathcal{S}(\mathfrak{n}_{G_P})'$ do not generally vanish on $\mathfrak{a}_P$, so $\mathcal{S}(\widetilde{\mathfrak{n}_{G_P})'|_{\mathfrak{a}_{G_P}}}$ is not simply $\mathcal{S}(\mathfrak{n}_{G_P})'$.

We describe the characters $\widetilde{\alpha'|_{\mathfrak{a}_{G_P}}}$ in $\mathcal{S}(\widetilde{\mathfrak{n}_{G_P})'}|_{\mathfrak{a}_{G_P}}$ explicitly. Recall that $\mathcal{S}(\mathfrak{n}_{G_P})' = \{\alpha_i' \,:\, i \neq m_k\}$.

Given i in the range $m_{k-1} < i < m_k$, let ℓ be defined by $i = m_{k-1} + \ell$. Then i is the ℓth element in the kth partition $\mathscr{I}_k = \{m_{k-1} < i \leq m_k\}$ of $\{1, \ldots, n\}$, with $1 \leq \ell \leq n_k - 1$. Using the explicit description of the matrix decomposition $H = H_{\mathfrak{a}_{G_P}} + H_{\mathfrak{a}_P}$ in A.3 and the explicit description of $\alpha_i'|_{\mathfrak{a}_{G_P}}$ in A.5, we can see that

$$\widetilde{\alpha_i'|_{\mathfrak{a}_{G_P}}} \;=\; \begin{pmatrix} \text{sum of first } \ell \text{ entries} \\ \text{in } k\text{th block} \end{pmatrix} \;-\; \ell \begin{pmatrix} \text{avg. of entries in} \\ k\text{th block} \end{pmatrix} .$$

Note that the first ℓ entries in the kth block are the entries in the kth block up to and including the ith entry of the whole matrix.

For example, for $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, and $H = \left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right)$,

$$\begin{aligned}
\widetilde{\alpha_1'|_{\mathfrak{a}_{G_P}}}(H) &= h_1 - \tfrac{1}{3}(h_1 + h_2 + h_3) = \tfrac{2}{3}h_1 - \tfrac{1}{3}h_2 - \tfrac{1}{3}h_3 \\
\widetilde{\alpha_2'|_{\mathfrak{a}_{G_P}}}(H) &= h_1 + h_2 - \tfrac{2}{3}(h_1 + h_2 + h_3) = \tfrac{1}{3}h_1 + \tfrac{1}{3}h_2 - \tfrac{2}{3}h_3 \\
\widetilde{\alpha_4'|_{\mathfrak{a}_{G_P}}}(H) &= h_4 - \tfrac{1}{2}(h_4 + h_5) = \tfrac{1}{2}h_4 - \tfrac{1}{2}h_5 \,,
\end{aligned}$$

and these three characters constitute the basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$ that is dual to $\mathcal{S}(\mathfrak{n}_{G_P})$.

We may also write $\widetilde{\alpha_i'|_{\mathfrak{a}_{G_P}}}$ as a linear combination of $\alpha_j \in \mathcal{S}(\mathfrak{n})$. For i with $m_{k-1} < i < m_k$, $1 \leq k \leq r+1$,

$$\widetilde{\alpha_i'|_{\mathfrak{a}_{G_P}}} \;=\; \sum_{j=1}^{n-1} c_j\,\alpha_j \quad \text{where} \quad c_j \;=\; \begin{cases} (j - m_{k-1})(m_k - i)/n_k & \text{if } m_{k-1} < j \leq i \\ (m_k - j)(i - m_{k-1})/n_k & \text{if } i < j < m_k \\ 0 & \text{else} \end{cases} .$$

Note that we do not need all the characters in $\mathcal{S}(\mathfrak{n})$ just those in $\mathcal{S}(\mathfrak{n}_{G_P})$, since, as we have already shown, $\mathcal{S}(\mathfrak{n}_{G_P})$ is a basis for $\mathfrak{a}_{G_P}^\vee$.

For example, for $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$, the basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$ that is dual to $\mathcal{S}(\mathfrak{n}_{G_P})$ consists of:

$$\widetilde{\alpha_1'|_{\mathfrak{a}_{G_P}}} = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \,, \quad \widetilde{\alpha_2'|_{\mathfrak{a}_{G_P}}} = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2 \,, \quad \widetilde{\alpha_4'|_{\mathfrak{a}_{G_P}}} = \tfrac{1}{2}\alpha_4 \,.$$

Now we write $\widetilde{\alpha_i'|_{\mathfrak{a}_{G_P}}}$ as a linear combination of $\alpha_j' \in \mathcal{S}(\mathfrak{n})'$. For i such that $m_{k-1} < i < m_k$, $1 \leq k \leq r+1$,

$$\widetilde{\alpha_i'|_{\mathfrak{a}_{G_P}}} = -\left(\tfrac{m_k - i}{n_k}\right)\alpha_{m_{k-1}}' + \alpha_i' - \left(\tfrac{i - m_{k-1}}{n_k}\right)\alpha_{m_k}' \,,$$

where $\alpha_{m_0}' = \alpha_0' = \alpha_{m_{r+1}}' = 0$. In particular, note that $\widetilde{\alpha_i'|_{\mathfrak{a}_{G_P}}} \neq \alpha_i'$, although both agree on $\mathfrak{a}_{G_P}$.

Returning to our example, $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$, we see

$$\widetilde{\alpha_1'|_{\mathfrak{a}_{G_P}}} = \alpha_1' - \tfrac{1}{3}\alpha_3' \,, \quad \widetilde{\alpha_2'|_{\mathfrak{a}_{G_P}}} = \alpha_2' - \tfrac{2}{3}\alpha_3' \,, \quad \widetilde{\alpha_4'|_{\mathfrak{a}_{G_P}}} = -\tfrac{1}{2}\alpha_3' + \alpha_4' \,.$$

A.6.2 Bases for $\widetilde{\mathfrak{a}_P^\vee}$

Now we describe bases for $\widetilde{\mathfrak{a}_P^\vee}$ corresponding to $\mathcal{S}(\mathfrak{n}_P)$ and $\mathcal{S}(\mathfrak{n}_P)'$. Based on our work with $\widetilde{\mathfrak{a}_{G_P}^\vee}$, we may hope that $\mathcal{S}(\mathfrak{n}_P)$ is a basis for $\widetilde{\mathfrak{a}_P^\vee}$, but this is not the case. In fact, elements of $\mathcal{S}(\mathfrak{n}_P)$ need not vanish on $\mathfrak{a}_{G_P}$ and so need not even be in $\widetilde{\mathfrak{a}_P^\vee}$. However, since $\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}$ is a basis for $\mathfrak{a}_P^\vee$ as shown in A.5, we know by the isomorphism $\mathfrak{a}_P^\vee \cong \widetilde{\mathfrak{a}_P^\vee}$ that $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}}$ is a basis for $\widetilde{\mathfrak{a}_P^\vee}$.

Explicitly $\mathcal{S}(\mathfrak{n}_P) = \{\alpha_{m_k} \,:\, 1 \le k \le r\}$, so the basis $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}}$ for $\widetilde{\mathfrak{a}_P^\vee}$ consists of

$$\widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}} \;=\; \begin{pmatrix}\text{avg. of entries}\\ \text{in } k\text{th block}\end{pmatrix} - \begin{pmatrix}\text{avg. of entries in}\\ (k+1)\text{th block}\end{pmatrix}\,, \quad 1 \le k \le r\,.$$

Note that Jorgenson and Lang (inaccurately) state that $\mathcal{S}(\mathfrak{n}_P)$ is a basis for $\mathfrak{a}_P^\vee$ and then go on to denote the elements of $\mathcal{S}(\mathfrak{n}_P)$ by $\alpha_{P,1}, \ldots, \alpha_{P,r}$. We choose to refer to the elements of $\mathcal{S}(\mathfrak{n}_P)$ as $\alpha_{m_1}, \ldots, \alpha_{m_r}$ to make it clear that they lie in $\mathcal{S}(\mathfrak{n}) = \{\alpha_1, \ldots, \alpha_{n-1}\}$. But we adopt the notation $\alpha_{P,k} = \widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}}$. Thus in our notation $\{\alpha_{P,k} \,:\, 1 \le k \le r\}$ is a basis for the copy of $\mathfrak{a}_P^\vee$ inside $\mathfrak{a}^\vee$, which is one way of interpreting what Jorgenson and Lang intended in the first place.

For example, with $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, $\mathcal{S}(\mathfrak{n}_P) = \{\alpha_3\}$, and the basis $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}}$ for $\widetilde{\mathfrak{a}_P^\vee}$ consists of

$$\alpha_{P,1}(H) \;=\; \widetilde{\alpha_3|_{\mathfrak{a}_P}}\left(\begin{array}{ccc|cc} h_1 & & & & \\ & h_2 & & & \\ & & h_3 & & \\ \hline & & & h_4 & \\ & & & & h_5 \end{array}\right) \;=\; \tfrac{1}{3}(h_1+h_2+h_3) \,-\, \tfrac{1}{2}(h_4+h_5)\,.$$

For $G = SL_{12}(\mathbb{C})$, $P = P^{4,3,3,2}$,

$$\begin{aligned} \alpha_{P,1} &= \widetilde{\alpha_4|_{\mathfrak{a}_P}} = \tfrac{1}{4}(h_1+h_2+h_3+h_4) \,-\, \tfrac{1}{3}(h_5+h_6+h_7) \\ \alpha_{P,2} &= \widetilde{\alpha_7|_{\mathfrak{a}_P}} = \tfrac{1}{3}(h_5+h_6+h_7) - \tfrac{1}{3}(h_8+h_9+h_{10}) \\ \alpha_{P,3} &= \widetilde{\alpha_{10}|_{\mathfrak{a}_P}} = \tfrac{1}{3}(h_8+h_9+h_{10}) - \tfrac{1}{2}(h_{11}+h_{12})\,. \end{aligned}$$

We may write these characters in terms of the basis $\mathcal{S}(\mathfrak{n})$ of simple characters in $\mathfrak{a}^\vee$. For $1 \le k \le r$,

$$\alpha_{P,k} \;=\; \widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}} \;=\; \frac{1}{n_k}\sum_{\ell=1}^{n_k} \ell\,\alpha_{m_{k-1}+\ell} \;+\; \frac{1}{n_{k+1}}\sum_{\ell=1}^{n_{k+1}-1}(n_{k+1}-\ell)\,\alpha_{m_k+\ell}\,.$$

In particular, notice that $\widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}} \neq \alpha_{m_k}$ as elements of $\mathfrak{a}^\vee$.

For example, for $G = SL_5(\mathbb{C})$, $P = P^{3,2}$,

$$\alpha_{P,1} = \widetilde{\alpha_3|_{\mathfrak{a}_P}} \;=\; \tfrac{1}{3}(\alpha_1 + 2\alpha_2 + 3\alpha_3) \,+\, \tfrac{1}{2}(\alpha_4)\,.$$

For $G = SL_{12}(\mathbb{C})$, $P = P^{4,3,3,2}$,

$$\begin{aligned} \alpha_{P,1} &= \widetilde{\alpha_4|_{\mathfrak{a}_P}} = \tfrac{1}{4}(\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4) \,+\, \tfrac{1}{3}(2\alpha_5+\alpha_6) \\ \alpha_{P,2} &= \widetilde{\alpha_7|_{\mathfrak{a}_P}} = \tfrac{1}{3}(\alpha_5+2\alpha_6+3\alpha_7) \,+\, \tfrac{1}{3}(2\alpha_8+\alpha_9) \\ \alpha_{P,3} &= \widetilde{\alpha_{10}|_{\mathfrak{a}_P}} = \tfrac{1}{3}(\alpha_8+2\alpha_9+3\alpha_{10}) \,+\, \tfrac{1}{2}(\alpha_{11})\,. \end{aligned}$$

We may also write these characters in terms of the dual basis $\mathcal{S}(\mathfrak{n})'$. For $1 \le k \le r$,

$$\alpha_{P,k} \;=\; \widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}} \;=\; -\big(\tfrac{1}{n_k}\big)\alpha'_{m_{k-1}} \,+\, \big(\tfrac{1}{n_k}+\tfrac{1}{n_{k+1}}\big)\alpha'_{m_k} \,-\, \big(\tfrac{1}{n_{k+1}}\big)\alpha'_{m_{k+1}}\,.$$

For example, for $G = SL_5(\mathbb{C})$, $P = P^{3,2}$,

$$\alpha_{P,1} \;=\; \widetilde{\alpha_3|_{\mathfrak{a}_P}} \;=\; \left(\tfrac{1}{3}+\tfrac{1}{2}\right)\alpha_3' \;-\; \left(\tfrac{1}{2}\right)\alpha_5' \;=\; \tfrac{5}{6}\alpha_3' \,.$$

For $G = SL_{12}(\mathbb{C})$, $P = P^{4,3,3,2}$,

$$\begin{aligned}
\alpha_{P,1} &= \widetilde{\alpha_4|_{\mathfrak{a}_P}} = \left(\tfrac{1}{4}+\tfrac{1}{3}\right)\alpha_4' - \tfrac{1}{3}\alpha_7' = \tfrac{7}{12}\alpha_4' - \tfrac{1}{3}\alpha_7' \\
\alpha_{P,2} &= \widetilde{\alpha_7|_{\mathfrak{a}_P}} = -\tfrac{1}{3}\alpha_4' + \left(\tfrac{1}{3}+\tfrac{1}{3}\right)\alpha_7' - \tfrac{1}{3}\alpha_{10}' = -\tfrac{1}{3}\alpha_4' + \tfrac{2}{3}\alpha_7' - \tfrac{1}{3}\alpha_{10}' \\
\alpha_{P,3} &= \widetilde{\alpha_{10}|_{\mathfrak{a}_P}} = -\tfrac{1}{3}\alpha_7' + \left(\tfrac{1}{3}+\tfrac{1}{2}\right)\alpha_{10}' = -\tfrac{1}{3}\alpha_7' + \tfrac{5}{6}\alpha_{10}'.
\end{aligned}$$

Note that, in fact, all we need are the characters in $\mathcal{S}(\mathfrak{n}_P)'$. We shall see below that $\mathcal{S}(\mathfrak{n}_P)'$ is a basis for $\widetilde{\mathfrak{a}_P^\vee}$.

Finally we consider a basis for $\widetilde{\mathfrak{a}_P^\vee}$ that is dual to $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}}$. A priori, this is $\widetilde{\mathcal{S}(\mathfrak{n}_P)'|_{\mathfrak{a}_P}}$; this is clear from the fact that $\mathcal{S}(\mathfrak{n}_P)'|_{\mathfrak{a}_P}$ is a basis for $\mathfrak{a}_P^\vee$ that is dual to $\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}$, as we have shown in A.5. . However, note that for $\alpha_{m_k}' \in \mathcal{S}(\mathfrak{n}_P)'$, and $H \in \mathfrak{a}_{G_P}$,

$$\alpha_{m_k}'(H) = (h_1 + \cdots + h_{m_1}) \,+\, \ldots \,+\, (h_{m_{k-1}+1} + \cdots + h_{m_k}) \;=\; 0 + \cdots + 0 \;=\; 0\,,$$

i.e. $\alpha_{m_k}'|_{\mathfrak{a}_{G_P}} \equiv 0$, so $\widetilde{\alpha_{m_k}'|_{\mathfrak{a}_P}} \equiv \alpha_{m_k}'$ on $\mathfrak{a}$, and indeed $\widetilde{\mathcal{S}(\mathfrak{n}_P)'|_{\mathfrak{a}_P}} = \mathcal{S}(\mathfrak{n}_P)'$.

A.7 Summary of Bases

We summarize the results of the preceding subsections in a table, for convenience. We use the abbreviated phrase "$i \neq m_k$", to mean "$1 \leq i \leq n-1$, $i \neq m_k$ for any k with $1 \leq k \leq r$," and similarly, whenever we write "$i = m_k$," we mean "$i = m_k$ for some k with $1 \leq k \leq r$."

Space	Basis		Dual Basis	
$\mathfrak{a}$	$\mathscr{B}$	$= \{H_i \,:\, 1 \leq i \leq n-1\}$	$\mathscr{B}'$	$= \{H_i' \,:\, 1 \leq i \leq n-1\}$
$\mathfrak{a}^\vee$	$\mathcal{S}(\mathfrak{n})$	$= \{\alpha_i \,:\, 1 \leq i \leq n-1\}$	$\mathcal{S}(\mathfrak{n})'$	$= \{\alpha_i' \,:\, 1 \leq i \leq n-1\}$
$\mathfrak{a}_{G_P}$	$\mathscr{B} \cap \mathfrak{a}_{G_P}$	$= \{H_i \,:\, i \neq m_k\}$	$\pi_{\mathfrak{a}_{G_P}}(\mathscr{B}' \cap \mathfrak{a}_{G_P})$	$= \{(H_i')_{\mathfrak{a}_{G_P}} \,:\, i \neq m_k\}$
$\mathfrak{a}_P$	$\pi_{\mathfrak{a}_P}(\mathscr{B} \cap \mathfrak{a}_P)$	$= \{(H_i)_{\mathfrak{a}_P} \,:\, i = m_k\}$	$\mathscr{B}' \cap \mathfrak{a}_P$	$= \{H_i' \,:\, i = m_k\}$
$\mathfrak{a}_{G_P}^\vee$	$\mathcal{S}(\mathfrak{n}_{G_P})\|_{\mathfrak{a}_{G_P}}$	$= \{\alpha_i\|_{\mathfrak{a}_{G_P}} \,:\, i \neq m_k\}$	$\mathcal{S}(\mathfrak{n}_{G_P})'\|_{\mathfrak{a}_{G_P}}$	$= \{\alpha_i'\|_{\mathfrak{a}_{G_P}} \,:\, i \neq m_k\}$
$\mathfrak{a}_P^\vee$	$\mathcal{S}(\mathfrak{n}_P)\|_{\mathfrak{a}_P}$	$= \{\alpha_i\|_{\mathfrak{a}_P} \,:\, i = m_k\}$	$\mathcal{S}(\mathfrak{n}_P)'\|_{\mathfrak{a}_P}$	$= \{\alpha_i'\|_{\mathfrak{a}_P} \,:\, i = m_k\}$
$\widetilde{\mathfrak{a}_{G_P}^\vee}$	$\mathcal{S}(\mathfrak{n}_{G_P})$	$= \{\alpha_i \,:\, i \neq m_k\}$	$\widetilde{\mathcal{S}(\mathfrak{n}_{G_P})'\|_{\mathfrak{a}_{G_P}}}$	$= \{\widetilde{\alpha_i'\|_{\mathfrak{a}_{G_P}}} \,:\, i \neq m_k\}$
$\widetilde{\mathfrak{a}_P^\vee}$	$\widetilde{\mathcal{S}(\mathfrak{n}_P)\|_{\mathfrak{a}_P}}$	$= \{\widetilde{\alpha_i\|_{\mathfrak{a}_P}} \,:\, i = m_k\}$	$\mathcal{S}(\mathfrak{n}_P)'$	$= \{\alpha_i' \,:\, i = m_k\}$

A.8 Orthogonality of Characters in $\mathfrak{a}^\vee$

Given $\alpha \in \mathcal{S}(\mathfrak{n}_P)$ and $\beta \in \mathcal{S}(\mathfrak{n}_{G_P})$, we do *not* expect α orthogonal to β nor α' orthogonal to β'.

For example, with $G = SL_5(\mathbb{C})$, $P = P^{3,2}$, $\alpha = \alpha_3$ and $\beta = \alpha_4$,

$$\langle \alpha, \beta \rangle \;=\; \langle \alpha_3, \alpha_4 \rangle \;=\; \langle H_3, H_4 \rangle \;=\; -1 \;\neq\; 0\,,$$

$$\langle \alpha', \beta' \rangle \;=\; \langle \alpha'_3, \alpha'_4 \rangle \;=\; \langle H'_3, H'_4 \rangle \;=\; \tfrac{3}{5} \;\neq\; 0\;.$$

See the A.3 for explicit descriptions of the matrices H_i and H'_i.

However, for all $\alpha \in \mathcal{S}(\mathfrak{n}_P)$ and $\beta \in \mathcal{S}(\mathfrak{n}_{G_P})$, we do have the following four orthogonality statements, all of which hold in $\mathfrak{a}^\vee$:

$$(1)\;\; \widetilde{\alpha|_{\mathfrak{a}_P}} \perp \beta\;, \qquad (2)\;\; \alpha' \perp \widetilde{\beta'|_{\mathfrak{a}_{G_P}}}\;, \qquad (3)\;\; \alpha' \perp \beta\;, \qquad (4)\;\; \alpha \perp \beta'\;.$$

Note that these orthogonality statements give us four ways to construct a basis for $\mathfrak{a}^\vee$ that is a disjoint union of (mutually orthogonal) bases for $\widetilde{\mathfrak{a}_P^\vee}$ and $\widetilde{\mathfrak{a}_{G_P}^\vee}$.

A.9 Failure to prove the orthogonality of ρ_{G_P} and ρ_P

After our discussion of bases for $\widetilde{\mathfrak{a}_P^\vee}$ and $\widetilde{\mathfrak{a}_{G_P}^\vee}$ in A.6 and constructing bases for $\mathfrak{a}^\vee$ that can be written as a disjoint union of (mutually orthogonal) bases for $\widetilde{\mathfrak{a}_P^\vee}$ and $\widetilde{\mathfrak{a}_{G_P}^\vee}$ in A.8, we may (naively) hope that a proof for the orthogonality of ρ_{G_P} and ρ_P can be obtained relatively quickly using these fairly abstract results.

However, this approach fails, for although ρ_{G_P} can easily be shown to lie in $\widetilde{\mathfrak{a}_{G_P}^\vee}$—since it is defined as the sum of relevant characters in $\mathcal{R}(\mathfrak{n}_{G_P})$, all such relevant characters are sums of simple characters in $\mathcal{S}(\mathfrak{n}_P)$, and the set $\mathcal{S}(\mathfrak{n}_{G_P})$ is a basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$—it is not straightforward to show that ρ_P lies in $\widetilde{\mathfrak{a}_P^\vee}$. As mentioned above, $\mathcal{R}(\mathfrak{n}_P)$ does not lie in $\widetilde{\mathfrak{a}_P^\vee}$, characters in $\mathcal{R}(\mathfrak{n}_P)$ are not expressible as linear combinations of elements in $\mathcal{S}(\mathfrak{n}_P)$, and $\mathcal{S}(\mathfrak{n}_P)$ does not form a basis for $\widetilde{\mathfrak{a}_P^\vee}$ in any case.

In retrospect, we should not be surprised at the failure of arguments neglecting to take into account the highly structured nature of the situation, in particular the facts that $\mathfrak{g}_{G_P}$ is a direct sum of copies of $\mathfrak{sl}_{n_k}(\mathbb{C})$, $1 \le k \le r+1$ and that ρ, ρ_{G_P}, and ρ_P reflect the structures of $\mathfrak{g}$ and $\mathfrak{g}_{G_P}$.

B The "half-trace" ρ

B.1 Formulas for the "half-trace" ρ

The "half-trace" of the regular representation of $\mathfrak{a}$ on $\mathfrak{n}$ is defined as

$$\rho \;=\; \tfrac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathfrak{n})} m_\alpha \alpha\;,$$

where m_α is the multiplicity of α, namely $m_\alpha = \dim_{\mathbb{R}}(\mathfrak{n}_\alpha)$. Note that for $G = SL_n(\mathbb{C})$, $m_\alpha = 2$ for all $\alpha \in \mathcal{R}(\mathfrak{n})$, so

$$\rho \;\overset{SL_n(\mathbb{C})}{=}\; \sum_{\alpha \in \mathcal{R}(\mathfrak{n})} \alpha\;.$$

We may write ρ in terms of the basis $\mathcal{S}(\mathfrak{n}) = \{\alpha_i \,:\, 1 \le i \le n-1\}$ of $\mathfrak{a}^\vee$, using the fact that that every $\alpha \in \mathcal{R}(\mathfrak{n})$ is of the form $\alpha = \alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$ for some $1 \le i < j \le n$.

$$\rho \;=\; \sum_{i=1}^{n-1} i(n-i)\alpha_i\;.$$

Using this and the formula for α_i in terms of the basis $\mathcal{S}(\mathfrak{n})'$ in A.3, we may also write ρ in terms of the basis $\mathcal{S}(\mathfrak{n})' = \{\alpha'_i \,:\, 1 \le i \le n-1\}$ of $\mathfrak{a}^\vee$, which is dual to $\mathcal{S}(\mathfrak{n})$,

$$\rho \;=\; 2 \sum_{\alpha' \in \mathcal{S}(\mathfrak{n})'} \alpha'\;.$$

B.2 That $\langle \rho, \alpha \rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n})$ and an analogous statement for ρ_{G_P}

First we show that, for $\alpha \in \mathcal{S}(\mathfrak{n})$, $\langle \rho, \alpha \rangle = 2$. This is obtained easily from the formula for ρ in terms of the basis $\mathcal{S}(\mathfrak{n})'$ for $\mathfrak{a}^\vee$ which is dual to $\mathcal{S}(\mathfrak{n})$; see B.1. Indeed, for any $\alpha = \alpha_j \in \mathcal{S}(\mathfrak{n})$,

$$\langle \rho, \alpha \rangle \;=\; \Big\langle 2 \sum_{\alpha'_i \in \mathcal{S}(\mathfrak{n})'} \alpha'_i,\ \alpha_j \Big\rangle \;=\; 2 \sum_{\alpha'_i \in \mathcal{S}(\mathfrak{n})'} \langle \alpha'_i, \alpha_j \rangle \;=\; 2\,.$$

We will show that an analogous result holds for ρ_{G_P}, namely: for any $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$, $\langle \rho_{G_P}, \alpha \rangle = 2$.

Note that the analogous statement for ρ_P is false; see B.3.

The key observation is that $\mathfrak{g}_{G_P} \cong \mathfrak{sl}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_{n_{r+1}}(\mathbb{C})$ and $\mathfrak{a}_{G_P}$ is the direct product of isomorphic copies of the Iwasawa $\mathfrak{a}$-parts $\mathfrak{a}_{\mathfrak{sl}_{n_k}}$ of the $\mathfrak{sl}_{n_k}(\mathbb{C})$ for $1 \le k \le r+1$. In order to avoid excessively cumbersome notation, we denote these isomorphic copies as $\mathfrak{a}_1, \ldots, \mathfrak{a}_{r+1}$, so that

$$\mathfrak{a}_{G_P} \;=\; \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_{r+1} \qquad \text{with} \quad \mathfrak{a}_k \cong \mathfrak{a}_{\mathfrak{sl}_{n_k}} \subset \mathfrak{sl}_{n_k}(\mathbb{C})\,.$$

The matrices in $\mathfrak{a}_k$ are block diagonal, with all the blocks except the kth block being zero, and the kth block being diagonal with trace zero.

Recall that

$$\{H_i \;:\; 1 \le i \le n-1,\ i \ne m_k \;\text{ for any } 1 \le k \le r\}$$

and

$$\{(H'_i)_{\mathfrak{a}_{G_P}} \;:\; 1 \le i \le n-1,\ i \ne m_k \;\text{ for any } 1 \le k \le r\}$$

are bases for $\mathfrak{a}_{G_P}$ that are dual to each other. See A.4 for explicit descriptions of the matrices in these bases. We obtain bases for $\mathfrak{a}_k$ as subsets of these bases for $\mathfrak{a}_{G_P}$. For fixed k, the sets

$$\mathscr{B}_k \;=\; \{H_i \;:\; m_{k-1} < i < m_k\} \quad \text{and} \quad \mathscr{B}'_k \;=\; \{(H'_i)_{\mathfrak{a}_{G_P}} \;:\; m_{k-1} < i < m_k\}$$

are bases for $\mathfrak{a}_k$ that are dual to each other. Note that the matrices in $\mathscr{B}_k$ and $\mathscr{B}'_k$ are block diagonal, with each block except for the kth block being a zero matrix and the kth block of being of the same form (but smaller size, namely $n_k \times n_k$ instead of $n \times n$) as the matrices in the bases $\mathscr{B} = \{H_i\}$ and $\mathscr{B}' = \{H'_i\}$, respectively, of $\mathfrak{a}$; see A.3.

Next we describe a basis for the isomorphic copy $\widetilde{\mathfrak{a}_k^\vee}$ of $\mathfrak{a}_k^\vee$ in $\mathfrak{a}^\vee$. Recall that $\mathcal{S}(\mathfrak{n}_{G_P})$ is a basis for the isomorphic copy $\widetilde{\mathfrak{a}_{G_P}^\vee}$ of $\mathfrak{a}_{G_P}^\vee$ inside $\mathfrak{a}^\vee$; see A.2 and A.6.

Since every $\alpha \in \mathcal{S}(\mathfrak{n}_{G_P})$ is in the form α_i, for $1 \le i \le n-1$ and $i \ne m_k$ for any $1 \le k \le r$, we may partition $\mathcal{S}(\mathfrak{n}_{G_P})$ as follows:

$$\mathcal{S}(\mathfrak{n}_{G_P}) \;=\; \bigsqcup_{1 \le k \le r+1} \mathcal{S}(\mathfrak{n}_{G_P})_k\,, \qquad \text{where} \quad \mathcal{S}(\mathfrak{n}_{G_P})_k \;=\; \{\alpha_i \;:\; m_{k-1} < i < m_k\}\,,$$

and $\mathcal{S}(\mathfrak{n}_{G_P})_k$ is a basis for $\widetilde{\mathfrak{a}_k^\vee} \subset \mathfrak{a}^\vee$.

Moreover, since $\mathcal{R}(\mathfrak{n}_{G_P})$ consists of all the characters $\alpha_{i,j}$ where i and j are in the same partition $\mathscr{I}_k = \{i \;:\; m_{k-1}+1 \le i \le m_k\}$ of indices, we can partition $\mathcal{R}(\mathfrak{n}_{G_P})$ correspondingly,

$$\mathcal{R}(\mathfrak{n}_{G_P}) \;=\; \bigsqcup_{1 \le k \le r+1} \mathcal{R}(\mathfrak{n}_{G_P})_k \qquad \text{where} \quad \mathcal{R}(\mathfrak{n}_{G_P})_k \;=\; \{\alpha_{i,j} \;:\; i,j \in \mathscr{I}_k\}\,,$$

and every character in $\mathcal{R}(\mathfrak{n}_{G_P})_k$ is a sum of elements of $\mathcal{S}(\mathfrak{n}_{G_P})_k$, since $\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$.

In fact, the structure of $\mathcal{S}(\mathfrak{n}_{G_P})_k$ inside $\mathcal{R}(\mathfrak{n}_{G_P})_k$ is the same as the structure of the simple characters inside the relevant characters for $\mathfrak{sl}_{n_k}(\mathbb{C})$. This motivates defining $\rho_k \in \widetilde{\mathfrak{a}_k^\vee}$ by

$$\rho_k \;=\; \sum_{\alpha\in\mathcal{R}(\mathfrak{n}_{G_P})_k} \alpha,$$

and aiming to show:

$$\rho_k \;=\; \sum_{\ell=1}^{n_k-1} \ell(n_k-\ell)\alpha_{m_{k-1}+\ell} \;=\; 2\sum_{\ell=1}^{n_k-1} \widetilde{\alpha'_{m_{k-1}+\ell}}\;.$$

The proof of the first part of this formula is very similar to the proof of the corresponding formula in B.1. To prove the second part of the formula, we need to write an arbitrary $\alpha_i \in \mathcal{S}(\mathfrak{n}_{G_P})_k$ in terms of a dual basis for $\widetilde{\mathfrak{a}_k^\vee}$ in $\mathfrak{a}^\vee$. Recall that for $\alpha_i \in \mathcal{S}(\mathfrak{n})$,

$$\alpha_i \;=\; -\alpha'_{i-1} \;+\; 2\alpha'_i \;-\; \alpha'_{i+1}\;,$$

where, in order to allow this formula to apply to α_1 and α_{n-1}, we define $\alpha'_0 = \alpha'_n = 0$. The isomorphism $\mathfrak{a}^\vee \cong \mathfrak{a}$ gives

$$H_i \;=\; -H'_{i-1} \;+\; 2H'_i \;-\; H'_{i+1}\;,$$

where $H'_0 = H'_n = \mathbb{0}_n$. Thus we expect that for fixed k and for $m_{k-1} < i < m_k$,

$$H_i \;=\; -(H'_{i-1})_{\mathfrak{a}_{G_P}} \;+\; 2(H'_i)_{\mathfrak{a}_{G_P}} \;-\; (H'_{i+1})_{\mathfrak{a}_{G_P}}\;.$$

Indeed, given the observation above about the matrices in $\mathscr{B}_k$ and $\mathscr{B}'_k$ being zero except on the kth block, where they are of the same form (but smaller size) as the matrices in $\mathscr{B}$ and $\mathscr{B}'$, respectively, the only thing to check is that the formula holds in the boundary cases $i = m_{k-1}+1$ and $i = m_k - 1$. These cases are easily verified, the key being that $H'_{m_{k-1}}$ and H'_{m_k} are both in $\mathfrak{a}_P$, so their projections to $\mathfrak{a}_{G_P}$ are zero. Thus we have the corresponding formula for characters in $\widetilde{\mathfrak{a}_k^\vee}$. For i in the range $m_{k-1} < i < m_k$,

$$\alpha_i \;=\; -\widetilde{\alpha'_{i-1}|_{\mathfrak{a}_{G_P}}} \;+\; 2\,\widetilde{\alpha'_i|_{\mathfrak{a}_{G_P}}} \;-\; \widetilde{\alpha'_{i+1}|_{\mathfrak{a}_{G_P}}}.$$

To simplify the notation, for fixed k, we let

$$\alpha'_{G_P,\ell} \;=\; \widetilde{\alpha'_i|_{\mathfrak{a}_{G_P}}}\;, \qquad \text{where } \; i = m_{k-1}+\ell.$$

Thus $\{\alpha'_{G_P,\ell} \,:\, 1 \le \ell \le n_k - 1\}$ is a basis for $\widetilde{\mathfrak{a}_k^\vee}$ that is dual to $\mathcal{S}(\mathfrak{n}_{G_P})_k$. Now we may prove the second part of the desired formula follows from the first part.

With this all in place, we may now show that ρ_{G_P} satisfies the desired relation. Indeed,

$$\rho_{G_P} \;=\; \sum_{k=1}^{r+1}\rho_k \;=\; 2\sum_{k=1}^{r+1}\sum_{\ell=1}^{n_k-1}\alpha'_{G_P,\ell} \;=\; 2\sum_{\alpha_i\in\mathcal{S}(\mathfrak{n}_{G_P})}\widetilde{\alpha'_i|_{\mathfrak{a}_{G_P}}}\;,$$

so for all $\alpha = \alpha_j \in \mathcal{S}(\mathfrak{n}_{G_P})$,

$$\langle\rho_{G_P},\alpha\rangle \;=\; 2\sum_{\alpha_i\in\mathcal{S}(\mathfrak{n}_{G_P})}\langle\widetilde{\alpha'_i|_{\mathfrak{a}_{G_P}}},\,\alpha_j\rangle \;=\; 2\,.$$

B.3 The analogous statement for ρ_P is false

Given the discussion in the previous section, we may wonder whether $\langle\rho_P,\alpha\rangle \stackrel{?}{=} 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n}_P)$. If so, then the fact that $\langle\rho,\alpha\rangle = 2$ for all $\alpha \in \mathcal{S}(\mathfrak{n}_P) \subset \mathcal{S}(\mathfrak{n})$ would imply that $\langle\rho_{G_P},\alpha\rangle$ is zero for all $\alpha \in \mathcal{S}(\mathfrak{n}_P)$, and this would perhaps provide another avenue for trying to prove the orthogonality of ρ_{G_P} and ρ_P. (See 3.2.) This approach is problematic for a variety of reasons. First we must remember that the relevant characters in $\mathcal{R}(\mathfrak{n}_P)$ cannot typically be written as sums of simple characters in $\mathcal{S}(\mathfrak{n}_P)$, and ρ_P cannot be written as a linear combination of elements of $\mathcal{S}(\mathfrak{n}_P)$. So showing that $\langle \rho_{G_P}, \alpha\rangle$ is 2 for all $\alpha \in \mathcal{S}(\mathfrak{n}_P)$ would not imply the orthogonality of ρ_{G_P} and ρ_P.

We may still wonder, based on apparent symmetries between $\mathfrak{a}_P$ and $\mathfrak{a}_{G_P}$, if $\langle \rho_P, \alpha\rangle$ is equal to 2 for all $\alpha \in \mathcal{S}(\mathfrak{n}_P)$. A quick look at an example shows that this is not true.

Consider $G = SL_5(\mathbb{C})$ and $P = P^{3,2}$. Then

$$\mathcal{R}(\mathfrak{n}_P) \;=\; \{\alpha_{1,4},\ \alpha_{1,5},\ \alpha_{2,4},\ \alpha_{2,5}\ ,\ \alpha_{3,4},\ \alpha_{3,5}\} \qquad \text{and} \qquad \mathcal{S}(\mathfrak{n}_P) \;=\; \{\alpha_3\}\ .$$

Thus for $\alpha = \alpha_3 = \alpha_{3,4} \in \mathcal{S}(\mathfrak{n}_P)$,

$$\begin{aligned} \langle \rho_P, \alpha\rangle \;&=\; \langle \alpha_{1,4}, \alpha_{3,4}\rangle \;+\; \langle \alpha_{1,5}, \alpha_{3,4}\rangle \;+\; \langle \alpha_{2,4}, \alpha_{3,4}\rangle \;+\; \langle \alpha_{2,5}, \alpha_{3,4}\rangle \;+\; \langle \alpha_{3,4}, \alpha_{3,4}\rangle \;+\; \langle \alpha_{3,5}, \alpha_{3,4}\rangle \\ &=\; 1 + 0 + 1 + 0 + 2 + 1 \;=\; 5 \;\neq\; 2\ . \end{aligned}$$

We conclude this section by carrying out, to the extent that it is possible, a discussion for ρ_P analogous to the discussion of ρ and ρ_{G_P} in the previous section, B.2. Recall that

$$\rho_P \;=\; \sum_{\alpha \in \mathcal{R}(\mathfrak{n}_P)} \alpha\ .$$

As noted above, the relevant characters $\alpha \in \mathcal{R}(\mathfrak{n}_P)$ are *not* (typically) sums of simple characters in $\mathcal{S}(\mathfrak{n}_P)$ nor even linear combinations of such simple characters. However, the restrictions $\alpha|_{\mathfrak{a}_P}$, for $\alpha \in \mathcal{R}(\mathfrak{n}_P)$ can be written as linear combinations of elements in $\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}$, and the corresponding linear extensions $\widetilde{\alpha|_{\mathfrak{a}_P}}$ to $\mathfrak{a}$ that are zero on $\mathfrak{a}_{G_P}$ can be written as linear combinations of elements in $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}} \subset \mathfrak{a}^\vee$. Thus, although it is not a priori clear that $\rho_P \in \widetilde{\mathfrak{a}_P^\vee}$, we can consider $\widetilde{\rho_P|_{\mathfrak{a}_P}}$, which is certainly in $\widetilde{\mathfrak{a}_P^\vee}$, and use what we know about bases for $\widetilde{\mathfrak{a}_P^\vee}$ to obtain formulas for $\widetilde{\rho_P|_{\mathfrak{a}_P}}$.

We begin with the observation that when we restrict relevant characters in $\mathcal{R}(\mathfrak{n}_P)$ to $\mathfrak{a}_P$, there is quite a bit of "collapsing" and what is left has a structure similar to the relevant characters for $\mathfrak{sl}_{r+1}(\mathbb{C})$.

Thus we partition $\mathcal{R}(\mathfrak{n}_P)$ as follows. For $1 \le k \le r$ and $k+1 \le k' \le r+1$, let $\mathcal{R}(\mathfrak{n}_P)_{k,k'}$ be all $\alpha_{i,j} \in \mathcal{R}(\mathfrak{n}_P)$ such that $m_{k-1} < i \le m_k$ and $m_{k'-1} < j \le m_{k'}$. Then

$$\mathcal{R}(\mathfrak{n}_P) \;=\; \bigsqcup_{k=1}^{r} \bigsqcup_{k'=k+1}^{r+1} \mathcal{R}(\mathfrak{n}_P)_{k,k'}\ .$$

Note that the cardinality of $\mathcal{R}(\mathfrak{n}_P)_{k,k'}$ is $n_k \cdot n_{k'}$. When restricting to $\mathfrak{a}_P$, $\mathcal{R}(\mathfrak{n}_P)_{k,k'}$ will "collapse," i.e. for all $\alpha_{i,j} \in \mathcal{R}(\mathfrak{n}_P)_{k,k'}$,

$$\alpha_{i,j}|_{\mathfrak{a}_P} \;\equiv\; \alpha_{m_k,m_{k'}}|_{\mathfrak{a}_P}\ , \qquad \text{so} \qquad \widetilde{\alpha_{i,j}|_{\mathfrak{a}_P}} \;\equiv\; \widetilde{\alpha_{m_k,m_{k'}}|_{\mathfrak{a}_P}}\ .$$

Denote this element of $\widetilde{\mathfrak{a}_P^\vee}$ by $\alpha_{P,k,k'}$. Explicitly:

$$\alpha_{P,k,k'} \;=\; \widetilde{\alpha_{m_k,m_{k'}}|_{\mathfrak{a}_P}} \;=\; \begin{pmatrix}\text{avg. of entries}\\ \text{in } k\text{th block}\end{pmatrix} - \begin{pmatrix}\text{avg. of entries in}\\ (k')\text{th block}\end{pmatrix}\ .$$

We can now write $\widetilde{\rho_P|_{\mathfrak{a}_P}}$ as

$$\widetilde{\rho_P|_{\mathfrak{a}_P}} \;=\; \sum_{\alpha \in \mathcal{R}(\mathfrak{n}_P)} \widetilde{\alpha|_{\mathfrak{a}_P}} \;=\; \sum_{k=1}^{r} \sum_{k'=k+1}^{r+1} \sum_{\alpha \in \mathcal{R}(\mathfrak{n}_P)_{k,k'}} \widetilde{\alpha|_{\mathfrak{a}_P}} \;=\; \sum_{k=1}^{r} \sum_{k'=k+1}^{r+1} n_k n_{k'}\, \alpha_{P,k,k'}\ .$$

Using the fact that

$$\alpha_{P,k,k'} \;=\; \alpha_{P,k} \;+\; \ldots \;+\; \alpha_{P,k'-1} \;,$$

arguments similar to those in the previous section, B.2, allow us to write $\widetilde{\rho_P|_{\mathfrak{a}_P}}$ in terms of elements $\alpha_{P,k}$ of the basis $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}}$, as follows:

$$\widetilde{\rho_P|_{\mathfrak{a}_P}} \;=\; \sum_{k=1}^{r} m_k(n-m_k)\,\alpha_{P,k} \;.$$

Recall from A.6 that we may write $\alpha_{P,k}$ in terms of the dual basis $\mathcal{S}(\mathfrak{n}_P)'$ for $\widetilde{\mathfrak{a}_P^\vee}$. Explicitly,

$$\alpha_{P,k} \;=\; \widetilde{\alpha_{m_k}|_{\mathfrak{a}_P}} \;=\; -\big(\tfrac{1}{n_k}\big)\alpha'_{m_{k-1}} \;+\; \big(\tfrac{1}{n_k}+\tfrac{1}{n_{k+1}}\big)\alpha'_{m_k} \;-\; \big(\tfrac{1}{n_{k+1}}\big)\alpha'_{m_{k+1}} \;.$$

Again, using arguments similar to those in B.2, we use this formula to obtain

$$\widetilde{\rho_P|_{\mathfrak{a}_P}} \;=\; \sum_{k=1}^{r}(n_k+n_{k+1})\,\alpha'_{m_k} \;.$$

In particular, it is clear that

$$\widetilde{\rho_P|_{\mathfrak{a}_P}} \;=\; \sum_{\alpha'\in\mathcal{S}(\mathfrak{n}_P)'} c_{\alpha'}\,\alpha' \;\neq\; 2\sum_{\alpha'\in\mathcal{S}(\mathfrak{n}_P)'}\alpha' \;,$$

contradicting what one might naively suppose if one were to draw a false parallel with ρ and ρ_{G_P}.

As it turns out, $\widetilde{\rho_P|_{\mathfrak{a}_P}} = \rho_P$. This is because ρ_P is orthogonal to all $\alpha \in \mathcal{S}(\mathfrak{n}_G)$, and $\mathcal{S}(\mathfrak{n}_G)$ is a basis for $\widetilde{\mathfrak{a}_{G_P}^\vee}$, thus $\rho_P \in \big(\widetilde{\mathfrak{a}_{G_P}^\vee}\big)^\perp = \widetilde{\mathfrak{a}_P^\vee}$. We emphasize that this is not obvious from working directly with ρ_P; the proof relies critically on the fact that $\langle\rho,\alpha\rangle = \langle\rho_{G_P},\alpha\rangle = 2$ for all $\alpha\in\mathcal{S}(\mathfrak{n}_{G_P})$, and this in turn is proven using the structural similarities between $\mathfrak{a}$ and $\mathfrak{a}_{G_P}$; see 3.2 and B.2.

Thus we can write ρ_P nicely in terms of the basis $\widetilde{\mathcal{S}(\mathfrak{n}_P)|_{\mathfrak{a}_P}} = \{\alpha_{P,k} \,:\, 1\le k\le r\}$ and its dual basis $\mathcal{S}(\mathfrak{n}_P)' = \{\alpha'_{m_k} \,:\, 1\le k\le r\}$, as follows:

$$\rho_P \;=\; \sum_{k=1}^{r} m_k(n-m_k)\,\alpha_{P,k} \;=\; \sum_{k=1}^{r}(n_k+n_{k+1})\,\alpha'_{m_k} \;.$$

From this we can show that, for $\alpha = \alpha_{m_\ell}\in\mathcal{S}(\mathfrak{n}_P)$, with $1\le \ell\le r$,

$$\langle\rho_P,\alpha\rangle \;=\; \sum_{k=1}^{r}(n_k+n_{k+1})\langle\alpha'_{m_k},\alpha_{m_\ell}\rangle \;=\; n_\ell+n_{\ell+1} \;,$$

which is not typically equal to 2.