Designing Poincaré series for number theoretic applications

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Outline

Poincaré Series

Construct an automorphic form by averaging over a discrete subgroup:

$$\text{Pé}_f(g) \;=\; \sum_{\gamma\in\Gamma} f(\gamma\cdot g)$$

E.g. f a test function on $\mathfrak{H} = SL_2(\mathbb{R})/SO(2)$, $\Gamma = SL_2(\mathbb{Z})$.

For applications, may want to choose data f that is not smooth or compactly supported.

Subconvexity application

- in dir. of RH, re: growth of ζ/L-functions on critical line
- Good, Diaconu-Goldfeld, Diaconu-Garrett, Letang
- data for Poincaré series not smooth or compactly supported
- hindsight: data is a solution to a differential equation

$$(\Delta - \lambda)u = \theta$$

where Δ is the Laplacian on $\mathfrak{H}$, $\lambda \in \mathbb{C}$, and θ is a distribution.

Differential equations viewpoint

Data for Poincaré series as solution to $(\Delta - \lambda)u = \theta$.

Advantages to viewpoint

- characterize Pé_u as solution to corresp. automorphic differential equation
- heuristically immediate automorphic spectral expansion!
- allows generalization to higher rank
- connection to eigenfunctions for pseudo-Laplacians

Lattice point counting identity (exact formula, not asymptotic)

- lattice points in symmetric space G/K, where G is complex

Eigenfunctions for pseudo-Laplacians

- mero. continuation of Eisenstein series (Colin de Verdiere)

Poincaré series from differential equation

free space	afc quotient space
$\mathfrak{H} = SL_2(\mathbb{R})/SO(2)$	$SL_2(\mathbb{Z})\backslash\mathfrak{H}$
higher rank: G/K	$\Gamma\backslash G/K$
$(\Delta - \lambda)u \;=\; \theta^{\text{free}}$	$(\Delta - \lambda)u \;=\; \theta^{\text{afc}}$
spectral expansion for u^{free} in zonal spherical fcns	spectral expansion for u^{afc} in cfms, Eis. series, residues

$$\text{Pé}_{u^{\text{free}}} \;=\; u^{\text{afc}}$$

geometric description afc spectral expansion

What is needed

- spherical transform of HC and Berezin
- global zonal spherical Sobolev theory
- gauges on groups
- automorphic spectral theory
- global automorphic Sobolev theory

Explicit examples

Simplest interesting higher rank examples as explicitly as possible.

Let G be a complex, semi-simple Lie group with finite center, K its maximal compact, Γ a discrete subgroup (e.g. $G = SL_n(\mathbb{C})$, $K = SU(n)$, $\Gamma = SL_n(\mathbb{Z}[i])$.)

- $\theta = \delta$ is Dirac delta
  - solution u is fundamental solution for $\Delta - \lambda$
  - app: lattice point counting
- $\theta = S_b$ is integrate-over-shell
  - app: pseudo-Laplacians

Outline

General set-up

- G: complex semi-simple Lie group, K: maximal compact
- $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ Iwasawa decomp.
- Σ: set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, Σ^+ positive roots
- $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$, m_α: multiplicity of α
- $\mathfrak{a}_{\mathbb{C}}^*$ the set of complex-valued linear functions on $\mathfrak{a}$

Spherical transform/inversion of H-C and Berezin

- $X = K\backslash G/K$, $\Xi = \mathfrak{a}^*/W \approx \mathfrak{a}_+$
- $\varphi_{\rho+i\xi}$, $\xi \in \mathfrak{a}^+$, zonal spherical function with Casimir eigenvalue $\lambda_\xi = -(|\xi|^2 + |\rho|^2)$
- spherical transform of bi-K-invariant f

$$\mathcal{F}f\,(\xi) \;=\; \int_G f(g)\,\overline{\varphi}_{\rho+i\xi}(g)\,dg$$

- inverse transform

$$\mathcal{F}^{-1}f \;=\; \int_\Xi f(\xi)\,\varphi_{\rho+i\xi}\,|\mathbf{c}(\xi)|^{-2}d\xi$$

where $\mathbf{c}(\xi)$ is the Harish-Chandra $\mathbf{c}$-function and $d\xi$ is the usual Lebesgue measure on $\mathfrak{a}^* \approx \mathbb{R}^n$.

Fundamental Solution

Consider $(\Delta - \lambda_z)^\nu \, u_z \; = \; \delta$, where $\nu \in \mathbb{N}$, $\lambda_z = z^2 - |\rho|^2, z \in \mathbb{C}$.

Global zonal spherical Sobolev theory $\Rightarrow \exists$ solution u_z, unique in Sobolev spaces.

Proposition

For integral $\nu \; > \; \dim(G/K)/2$, u_z *is a continuous left-*K*-invariant function on* G/K *with the following spectral expansion:*

$$u_z(g) \;\; = \;\; \int_{\Xi} \frac{(-1)^\nu}{(|\xi|^2 + z^2)^\nu} \;\; \varphi_{\rho + i\xi}(g) \, |\mathbf{c}(\xi)|^{-2} \, d\xi$$

NB: The condition on ν is for uniform pointwise convergence. In general, the spectral expansion converges in the Sobolev sense.

Explicit formula for fundamental solution

Theorem

For an integer $\nu > \dim(G/K)/2 = n/2 + d$*, where* d *is the number of positive roots, counted without multiplicities, and* $n = \dim(\mathfrak{a})$ *is the rank,* $u_z(a)$ *can be expressed in terms of a K-Bessel function*

$$\prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\log a)} \cdot \left(\frac{|\log a|}{2z} \right)^{\nu - d - n/2} \cdot K_{\nu - d - n/2}(z|\log a|)$$

In the odd rank case, with $\nu = d + \frac{n+1}{2}$*,*

$$u_z(a) = \prod_{\alpha \in \Sigma^+} \frac{\alpha(\log a)}{2 \sinh(\alpha(\log a))} \cdot \frac{e^{-z|\log a|}}{z}$$

Rank one fundamental solution

For $G = SL_2(\mathbb{C})$, the continuity is visible,

$$u_z(a_r) \;=\; \frac{r\,e^{-(2z-1)r}}{(2z-1)\ \sinh r} \quad \text{where} \;\; a_r \;=\; \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$$

Integrating along shells

Consider $(\Delta - \lambda_z)^\nu w_z = S_b$, where S_b is the distribution that integrates a function over the shell:

$$K \cdot \{a = \exp(H) : H \in \mathfrak{a}_+ \text{ with } |H| = b\} \cdot K / K$$

G=$SL_2(\mathbb{C})$: spherical shell of radius b in hyperbolic 3-space

As before,

- $\exists$ solution w_z, unique in global zonal spherical Sobolev spaces
- spherical inversion $\Rightarrow$ integral representation in terms of $\mathcal{F}S_b$
- uniform pointwise convergence for ν sufficiently large

Eigenfunctions of pseudo-Laplacians: *weaker* convergence desired:

- H^1-convergence (Sobolev topology, index 1)
- efcns for the Friedrichs extension of (a restriction of) the Laplacian lie in the domain of the Fr. ext'n $\subset H^1(X)$
- $\nu = 1 \Rightarrow H^1$-convergence, regardless of $\dim G/K$

Explicit formula for solution

Theorem

For $\nu > (n+2d+1)/4$, *the solution* $w_z(a)$ *is*

$$\int_{|H|=b} \left(\frac{|\log a - H|}{z}\right)^{\nu - n/2} K_{\nu-1/2}(z|\log a - H|) \prod_{\alpha \in \Sigma^+} \frac{\sinh(\alpha(H))}{\sinh(\alpha(\log a))} dH$$

In particular, when $n = \dim \mathfrak{a}^*$ *is odd,* $w_z(a)$ *is*

$$\int_{|H|=b} \frac{P_{\nu - \frac{n+1}{2}}(z|\log a - H|)\, e^{-z|\log a - H|}}{z^{2\nu - n}} \prod_{\alpha \in \Sigma^+} \frac{\sinh(\alpha(H))}{\sinh(\alpha(\log a))} dH$$

where $P_\ell(x)$ *is a degree* ℓ *polynomial, with explicit coefficients.*

Rank one solution

For $G = SL_2(\mathbb{C})$, with $\nu = 1$, ensuring H^1-convergence,

$$w_z(a_r) \;=\; \frac{-\sinh(b)}{z\sinh(r)} \cdot \begin{cases} e^{-2bz}\,\sinh(2rz) & \text{if } r < b \\ \sinh(2bz)\,e^{-2rz} & \text{if } r > b \end{cases}$$

and, with $\nu = 2$, ensuring uniform pointwise convergence,

$$w_z(a_r) \;=\; \frac{2\sinh(b)}{z^3\sinh(r)} \cdot \begin{cases} e^{-2bz}\,\big((1+2bz)\cosh(2rz) - 2rz\sinh(2rz)\big) \\ \big((1+2rz)\cosh(2bz) - 2bz\sinh(2bz)\big)e^{-2rz} \end{cases}$$

Outline

Winding-up

Average over Γ, discrete subgroup of G:

$$\text{Pé}_{\mathfrak{f}}(g) \;=\; \sum_{\gamma\in\Gamma} \mathfrak{f}(\gamma\cdot g)$$

Example:

- Pé_{u_z} is solution to automorphic PDE: $(\Delta-\lambda_z)^{\nu}u_z^{\mathsf{afc}}=\delta^{\mathsf{afc}}$
- automorphic solution has automorphic spectral expansion
- lattice point counting in higher rank symmetric spaces

Integrating along shells

Theorem

If the solution w_z is of sufficient rapid decay, the Poincaré series Pé_{v_z} converges absolutely and uniformly on compact sets, to a continuous function of moderate growth, square-integrable modulo Γ. Moreover, it has an automorphic spectral expansion, converging uniformly pointwise:

$$\int_{\Xi}^{\oplus} \frac{\frac{\pi^+(\rho)}{\pi^+(-i\xi)} \left(\int_{|H|=b} e^{-i\langle \xi, H\rangle} \prod \sinh(\alpha H)\, dH\right) \overline{\Phi}_{\xi}(x_0) \cdot \Phi_{\xi}}{(-1)^{\nu} \ (|\xi|^2 + z^2)^{\nu}}$$

where $\{\Phi_\xi\}$ denotes a suitable spectral family of spherical automorphic forms (cusp forms, Eisenstein series, and residues of Eisenstein series) and $\lambda_\xi = -(|\xi|^2 + |\rho|^2)$ is the Casimir eigenvalue of Φ_ξ.

Rank one: $G = SL_2(\mathbb{C})$, $\Gamma = SL_2(\mathbb{Z}[i])$, $\nu = 2$

$\text{Pé}_z(g) = 2\sinh(b)/z^3 \times$

$$\Bigg(\sum_{\sigma(\gamma g)<b} \frac{\big((1+2bz)\cosh(2\sigma(\gamma g)z) - 2\sigma(\gamma g)z\sinh(2rz)\big)e^{-2bz}}{\sinh(\sigma(\gamma g))}$$

$$+ \sum_{\sigma(\gamma g)>b} \frac{\big((1+2\sigma(\gamma g)z)\cosh(2bz) - 2bz\sinh(2bz)\big)e^{-2\sigma(\gamma g)z}}{\sinh(\sigma(\gamma g))} \Bigg)$$

where $\sigma(g)$ is the geodesic distance from gK to $x_0 = 1 \cdot K$.

Spectral expansion:

$$\text{Pé}_z = \sum_{f\ GL_2\ \text{cfm}} \frac{\sin(2bt_f) \cdot \overline{f}(x_0) \cdot f}{2t_f \sinh(b)(t_f^2 + z^2)^2} + \frac{\overline{\Phi}_0(x_0) \cdot \Phi_0}{(z^2 - \frac{1}{4})^2}$$

$$+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sin(2bt) \cdot E_{\frac{1}{2}-it}(x_0) \cdot E_{\frac{1}{2}+it}}{2t\sinh(b)(t^2+z^2)^2}\, dt$$

where $-(t_f^2 + \frac{1}{4})$ and $-(t^2 + \frac{1}{4})$ are the eigenvalues of f and $E_{\frac{1}{2}+it}$.

Convergence of Poincaré series?

Regardless of the convergence of the Poincaré series, the solution w_z^{afc} to the automorphic differential equation

- exists
- is unique in global automorphic Sobolev spaces
- has the given spectral expansion, converging in a global automorphic Sobolev space.

If desired, uniform pointwise convergence of the spectral expansion can be obtained by choosing ν sufficiently large.

The difficulty, even in the simplest possible higher rank cases, namely G complex of odd rank, of ascertaining whether w_z is of sufficiently rapid decay along the walls of the Weyl chambers, where $\prod \sinh(\alpha(\log a))$ blows up, is reason to question whether the explicit "geometric" Poincaré series representation of w^{afc} is actually needed in a given application or whether the automorphic spectral expansion suffices.

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