Eigenvalues of Pseudo-Laplacians and Compact Periods of Eisenstein Series

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History

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- RH within reach?!
- Hejhal: Haas' methods flawed
- Hejhal (1981), Colin de Verdière (1981, 1983)
- Garrett, Bombieri (current)
- D (current)

Pseudo-Laplacians permit non-smoothness in eigenfunctions

Let $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$, $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

- Eigenfunctions for Δ must be smooth.
- Eigenfunctions for pseudo-Laplacians not nec. smooth.
  - CdV: mero. cont. of Eis. series
  - $\widetilde{\Delta}_a$ on Lax-Phillips space
  - Truncations $\wedge^a E_s$ with $c_P(E_s)(ia) = a^s + c_s a^{1-s} = 0$, $c_s = \xi(2s-1)/\xi(2s)$, are eigenfunctions.

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- Construct $\widetilde{\Delta}_\omega$ to overlook non-smoothness at corner $\omega = e^{2\pi i/6}$ of fundamental domain.
  - Legitimize Haas' purported eigenvalues?

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Using global automorphic Sobolev theory:

- Construct solutions to distributional differential equation using automorphic spectral expansions.
- But **no solution will lie in the domain of** $\widetilde{\Delta}_\omega$.
- Discrete spectrum of $\widetilde{\Delta}_\omega$ is **empty**.

Solutions are not Eigenfunctions

Global Automorphic Sobolev Spaces:

- For $\ell \geqslant 0$, $H^\ell(X)$ is the closure of $C_c^\infty(X)$ with respect to the topology induced by $\langle f, g \rangle_\ell = \langle (1-\Delta)^\ell f, g \rangle_{L^2}$.
- $H^{-\ell}(X)$ is the Hilbert space dual of $H^\ell(X)$.
- $H^\ell(X) \subset H^{\ell-1}(X)$ and $(1-\Delta) : H^\ell(X) \xrightarrow{\sim} H^{\ell-2}(X)$

Dirac delta distribution: $\delta \in H^\ell(X)$ only for $\ell < -1$. Thus, solutions of $(\Delta - \lambda)u = \delta$ lie in $H^\ell(X)$ only for $\ell < 1$.

An unbounded operator on a Hilbert space $\mathcal{H}$ is a linear map from a subspace (the **domain** of the operator) to $\mathcal{H}$.

- Restrictions of Δ, e.g. Δ_ω, unbdd ops on $L^2(X)$.
- By Friedrichs' construction, $\mathrm{Dom}(\widetilde{\Delta}_\omega) \subset H^1(X)$.

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  1. Existence of $\widetilde{\Delta}_\theta$-eigenvalue $\lambda_w = w(w-1)$ with $\text{Re}(w) = \frac{1}{2}$ does imply vanishing of zeta function at w.
  2. Interlacing of $\widetilde{\Delta}_\theta$-eigenvalues with zeros of constant term at fixed height.

Automorphic Spectral Expansions

Analogue of Fourier inversion: automorphic spectral expansion in terms of eigenfunctions for Laplacian.

Example: $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, $\Delta = y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$

$$v \overset{\text{Sob}}{=} \sum_F \langle v, F\rangle \cdot F + \langle v, \Phi_0\rangle \cdot \Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \langle v, E_s\rangle \cdot E_s \, ds$$

- F ranges over an orthonormal basis of cusp forms
- Φ_0 is the constant automorphic form with unit L^2-norm
- E_s is the real analytic Eisenstein series

Can view such formulas as "pre-trace formulas."

Constructing the Pseudo-Laplacian

Let Θ be a compactly supported distribution on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$.

$$\Theta \;=\; \sum_F \Theta(\overline{F})\cdot F + \Theta(\overline{\Phi_0})\cdot\Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{1-s})\cdot E_s\, ds$$

Following Colin de Verdiere (1983),

$$\theta \;=\; \mathrm{Proj}_{nc}\Theta \;=\; \Theta(\overline{\Phi_0})\cdot\Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{1-s})\cdot E_s\, ds$$

Restrict Δ to $L^2_{nc}(X)\cap C^\infty_c(X)\cap \ker(\theta)$, and let $\widetilde{\Delta}_\theta$ be its Friedrichs extension. Then, for u in the domain of $\widetilde{\Delta}_\theta$,

$$(\widetilde{\Delta}_\theta - \lambda_w)u = 0 \qquad\Longleftrightarrow\qquad (\Delta-\lambda_w)u = (\mathrm{const})\cdot\theta$$

Vanishing of Periods

Theorem (Bombieri-Garrett)

Let θ, Δ_θ, and $\widetilde{\Delta}_\theta$ be as above. Suppose θ lies in $H^{-1}(X)$ and θ is real, in the sense that $\theta(\overline{\varphi}) = \overline{\theta(\varphi)}$ for all $\varphi \in C_c^\infty(X)$. Then the compact period θE_w vanishes when $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$.

Note

Hardy-Littlewood 1918 $\Rightarrow \theta = \mathrm{Proj}_{\mathrm{nc}} \delta_\omega^{\mathrm{afc}}$ satisfies the hypotheses. Here: $\theta E_s = \zeta_{\mathbb{Q}(\omega)}(s)/\zeta(2s) = \zeta(s)L(s,\chi)/\zeta(2s)$.

Corollary

Let $\theta = \mathrm{Proj}_{\mathrm{nc}} \delta_\omega^{\mathrm{afc}}$. If $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$, then $\zeta_{\mathbb{Q}(\omega)}(w) = 0$.

How Many Zeros?

With $\lambda_w = w(w-1)$ an eigenvalue for $\widetilde{\Delta}_\theta$.

Have shown:

- w's on the critical line $\subset$ zeros of zeta function

Hope: w's account for

- **all** zeros of zeta? (No: Epstein zetas.)
- **many** zeros of zeta? (Not clear.)

If Montgomery's Pair Correlation Conjecture is true,

- at most a **positive fraction** of zeros of zeta,
- because they **interlace** with zeros of $c_P(E_s)(ia) = a^s + c_s a^{1-s}$ on the critical line.

Lax-Phillips Discretization

Lax-Phillips space: for $a \gg 1$

$$L^2(X)_a \;=\; \{f \in L^2(X) : c_P f(z) = 0 \text{ when } \operatorname{Im}(z) \geqslant a\}$$

Let $\Delta_a = \Delta|_{L^2(X)_a \cap C_c^\infty(X)}$ and $\widetilde{\Delta}_a$ its Friedrichs extension.

$L^2(X)_a$ decomposes discretely:

- Orthogonal basis of $\widetilde{\Delta}_a$-eigenfunctions:
  - cuspforms
  - truncated Eisenstein series $\wedge^a E_s$ with $a^s + c_s a^{1-s} = 0$.
- Parametrize eigenvalues of non-cuspidal discrete spectrum: $\{\lambda_{s_j} = s_j(s_j - 1) : a^{s_j} + c_{s_j} a^{1-s_j} = 0\}_{0 \leqslant j \in \mathbb{Z}}$

Key: Eigenfunctions for $\widetilde{\Delta}_\theta$ lie in the non-cuspidal part of $L^2(X)_a$, so have expansions in terms of $\wedge^a E_s$'s.

Interlacing

Theorem (Bombieri-Garrett, Interlacing)

With Θ, θ, $\widetilde{\Delta}_\theta$ and $\widetilde{\Delta}_a$ as above, if a is chosen so that (1) $\mathrm{supp}(\Theta)$ lies below $\mathrm{Im}(z) = a$ and (2) $\theta E_{s_j} \neq 0$ for any j, then, between any two adjacent parameters s_{j_1} and s_{j_2} on the critical line there is at most one parameter w corresponding to an eigenvalue $\lambda_w = w(w-1)$ of $\widetilde{\Delta}_\theta$.

Corollary

Let $\theta = \mathrm{Proj}_{\mathrm{nc}}\delta_\omega^{\mathrm{afc}}$. Under Montgomery's Pair Correlation Conjecture, at most a proper fraction of the $\widetilde{\Delta}_\theta$-eigenvalue parameters appear among the zeros of the Riemann zeta function.

GL_3 Automorphic L-functions

- GL_3 automorphic L-functions arise as compact periods of GL_3 cuspidal data Eisenstein series.
- Theorem (D.) Vanishing of compact periods of GL_3 cuspidal data Eisenstein series at w-values on the critical line corresponding to eigenvalues (if any) of suitable pseudo-Laplacian.
- Theorem (D.) Interlacing with discrete spectrum of pseudo-Laplacian on Lax-Phillips space.
- H^{-1}-condition on period distribution: condition on the second moment of period.
- Given suitable moment bound, prove vanishing of GL_3 automorphic L-functions?

Quaternion Algebra Zeta Functions

Zeta functions for quaternion algebras also arise as compact periods of Eisenstein series.

Quaternion algebra B over k, split over quadratic ℓ/k,

- H is copy of $B^\times$ in $G = GL_2(\ell)$
- $\Theta_H f \;=\; \int_{ZH_k \backslash H_{\mathbb{A}_k}} f(h)\, dh$
- $\Theta_H\, E_s \;=\; (\text{const}) \times \frac{\xi_B(2s)}{\xi_\ell(2s)}$

Vanishing and interlacing should be accessible using similar methods.

Thank you for your attention!