Zeros of Zeta Functions and Eigenvalues of Pseudo-Laplacians

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Backstory

- Haas (1977) Zeros of zeta appear among parameter values $\{s : \lambda_s = s(s-1)\}$ for purported eigenvalues λ_s of Δ on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$
- RH within reach?!

Backstory

- Haas (1977) Zeros of zeta appear among parameter values $\{s : \lambda_s = s(s-1)\}$ for purported eigenvalues λ_s of Δ on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$
- RH within reach?!
- Hejhal: Haas' methods flawed
- Hejhal (1981), Colin de Verdière (1981, 1983)
- Garrett, Bombieri (current)

Zeros of zeta functions and eigenvalues of self-adjoint operators

Goal (following Hilbert and Polya): produce zeros of zeta functions (or other compact periods) among parameters w for eigenvalues $\lambda_w = w(w-1)$ of self-adjoint operators.

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Theorem (Rough Statement)

If $\lambda_w = w(w-1)$ is an eigenvalue of a (carefully constructed) self-adjoint operator, "$\widetilde{\Delta}_\theta$", then the period θE_w vanishes when w is on the critical line, i.e.

$$\{w \in \tfrac{1}{2} + i\mathbb{R} : \lambda_w = w(w-1) \text{ is an eigenvalue for } \widetilde{\Delta}_\theta\}$$

$$\subset \quad \{s : \theta E_s = 0\}$$

Zeros of Dedekind Zeta Function

In particular: $\theta = \delta_{z_0}^{\mathrm{afc}}$, with $z_0 = \omega$.

- Then $\theta E_s = \zeta_{\mathbb{Q}(\omega)}(s)/\zeta(2s)$
- In critical strip: $\theta E_s = 0 \iff \zeta_{\mathbb{Q}(\omega)} = 0$
- Construct (?) suitable "$\widetilde{\Delta}_\theta$" with non-empty (large!?) discrete spectrum

$\overset{\text{would}}{\Rightarrow}$ Large subset of zeros of $\zeta_{\mathbb{Q}(\omega)}$ on the critical line!

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Retrospect:

- Haas' error: failure to distinguish between Δ and "$\widetilde{\Delta}_\theta$"
- Zeros of $\zeta_{\mathbb{Q}(\omega)}(s) = \zeta(s)L(s,\chi)$ appeared among the parameter values for his purported eigenvalues of Δ.

What is $\widetilde{\Delta}_\theta$?

The operator $\widetilde{\Delta}_\theta$ will be

- the **Friedrichs extension** (necessarily self-adjoint)
- of a suitable **restriction** Δ_θ
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Given θ, want to choose Δ_θ such that:

$$(\widetilde{\Delta}_\theta - \lambda_w)u = 0 \quad \overset{??}{\Longleftrightarrow} \quad (\Delta - \lambda_w)u = (\text{const}) \cdot \theta$$

- Use "engineering math" to find solutions!
- But such solutions may not lie in the domain of $\widetilde{\Delta}_\theta$!
- For example, $\theta = \delta_{z_0}^{\text{afc}}$ (CdV)

To clarify, need details of Friedrichs extension and global automorphic Sobolev theory.

Simplest Case: δ on $\mathbb{R}$

Here $\Delta = \frac{d^2}{dx^2}$, $\lambda_w = 4\pi^2 w^2$, $\theta = \delta$, so $(\Delta - \lambda_w)u = \theta$ is

$$\left(\frac{d^2}{dx^2} - 4\pi^2 w^2\right)u \;=\; \delta$$

Apply a Fourier transform to both sides:

$$(-4\pi^2\xi^2 - 4\pi^2 w^2)\mathcal{F}(u) \;=\; \mathcal{F}(\delta) = 1$$

Division gives Fourier coefficients for u. Fourier inversion:

$$u(x) \;=\; \int_{\mathbb{R}} \mathcal{F}u(\xi)\, e^{2\pi i \xi x}\, d\xi \;=\; \frac{-1}{4\pi^2}\int_{\mathbb{R}} \frac{e^{2\pi i \xi x}}{\xi^2 + w^2}\, d\xi$$

$$= \; \frac{-e^{2\pi w|x|}}{4\pi w} \qquad (\mathrm{Re}(w) > 0)$$

Compactly supported θ on $\mathbb{R}$

For θ compactly supported: $(\frac{d^2}{dx^2} - 4\pi^2 w^2)u \;=\; \theta.$

$$u(x) \;=\; \frac{-1}{4\pi^2} \int_{\mathbb{R}} \frac{\theta(\overline{\psi}_\xi)\, e^{2\pi i \xi x}}{\xi^2 + w^2}\, d\xi \;=\; \int_{\mathbb{R}} \frac{\langle \theta, \psi_\xi \rangle \cdot \psi_\xi}{\lambda_\xi - \lambda_w}$$

where $\psi_\xi(x) = e^{2\pi i x \xi}$, $\lambda_\xi = -4\pi^2 \xi^2$.

The issue of convergence is non-trivial, and Sobolev theory provides a robust framework for addressing it.

Sobolev Spaces on $\mathbb{R}$

For ℓ a positive integer:

- Inner product on C_c^∞

$$\langle \varphi_1, \varphi_2 \rangle_\ell = \langle (1-\Delta)^\ell \varphi_1, \varphi_2 \rangle_{L^2}$$

- H^ℓ is completion of C_c^∞
- $H^{-\ell}$ is its Hilbert space dual

Notice

- $H^0 = L^2$
- $H^\ell \hookrightarrow H^{\ell-1}$ for all ℓ

Laplacian and Spectral Transform

Hilbert space isomorphisms:

- $(1-\Delta) : H^\ell \to H^{\ell-2}$
- $\mathcal{F} : H^\ell \to V^\ell$, a weighted L^2 space on the spectral side

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow[\approx]{(1-\Delta)} & H^\ell & \xrightarrow[\approx]{(1-\Delta)} & H^{\ell-2} & \xrightarrow[\approx]{(1-\Delta)} & \cdots \\
 & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \\
\cdots & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^\ell & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{\ell-2} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & \cdots
\end{array}
$$

where $\lambda_\xi = -4\pi^2\xi^2$ is the Δ-eigenvalue of $\psi_\xi = e^{2\pi i x \xi}$.

Justifying Engineering Math

$$\begin{array}{ccccccc}
\cdots & \xrightarrow[\approx]{(1-\Delta)} & H^{-\ell+2} & \xrightarrow[\approx]{(1-\Delta)} & H^{-\ell} & \xrightarrow[\approx]{(1-\Delta)} & \cdots \\
 & & \mathcal{F} \downarrow \approx & & \mathcal{F} \downarrow \approx & & \\
\cdots & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{-\ell+2} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{-\ell} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & \cdots
\end{array}$$

For $\theta \in H^{-\ell}$, there is unique u, satisfying $(\Delta - \lambda_w)u = \theta$. Further, u lies in $H^{-\ell+2}$, and has spectral expansion

$$u = \int_\Xi \frac{\langle \theta, \psi_\xi \rangle \cdot \psi_\xi}{\lambda_\xi - \lambda_w} \, d\xi = \frac{-1}{4\pi^2} \int_\mathbb{R} \frac{\theta(\overline{\psi}_\xi) \cdot e^{2\pi i \xi x}}{\xi^2 + w^2} \, d\xi$$

converging in $H^{-\ell+2}$. (And for $\ell > k + 1/2$, $H^\ell \hookrightarrow C^k$.)

Automorphic Spectral Theory

Analogue of Fourier inversion: automorphic spectral expansion in terms of eigenfunctions for Laplacian.

Example: $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, $\Delta = y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$

$$v \overset{L^2}{=} \sum_F \langle v, F\rangle \cdot F + \langle v, \Phi_0\rangle \cdot \Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \langle v, E_s\rangle \cdot E_s \, ds$$

- F ranges over an orthonormal basis of cusp forms
- Φ_0 is the constant automorphic form with unit L^2-norm
- E_s is the real analytic Eisenstein series

Abbreviate/generalize: $v = \int_\Xi^\oplus \langle v, \Phi_\xi\rangle \cdot \Phi_\xi \, d\xi$

Global Automorphic Sobolev Theory

- Compactly supported $\theta \in H^{-\ell}$ for some ℓ.
- Existence and uniqueness of solutions of $(\Delta - \lambda_w)u = \theta$.
- Spectral expansion of solution, converging in the $H^{-\ell+2}$-topology:

$$u = \int_{\Xi}^{\oplus} \frac{\langle \theta, \Phi_\xi \rangle \cdot \Phi_\xi}{\lambda_\xi - \lambda_w} \, d\xi$$

- A global automorphic Sobolev embedding theorem (for $\ell > k + \dim(X)/2$, $H^\ell \hookrightarrow C^k$).

Connection to Eigenvalue Question

- Expect to have correspondence:

$$(\text{sol'ns of } (\Delta - \lambda_w)u = (\text{const.})\theta) \leftrightarrow (\text{eig-fcns for "}\widetilde{\Delta}_\theta\text{"})$$

Connection to Eigenvalue Question

- Expect to have correspondence:

$$(\text{sol'ns of } (\Delta - \lambda_w)u = (\text{const.})\theta) \leftrightarrow (\text{eig-fcns for ``}\widetilde{\Delta}_\theta\text{''})$$

- Have a complete description of the solutions that lie in global automorphic Sobolev spaces.
- Turns out: such solutions are **not** always eigenfunctions.
- Issue is that solutions may not lie in the **domain** of the Friedrichs extension.

Friedrichs Extension

An **unbounded operator** on a Hilbert space is a linear map from a subspace (the **domain**) to the Hilbert space.

The Laplacian Δ, restricted to a suitable dense subspace of $L^2(X)$, like $C_c^\infty(X)$, is a densely defined nonpositive symmetric unbounded operator on $L^2(X)$.

Friedrichs' general construction gives a **self-adjoint extension**, which depends on the choice of domain.

Domain of Restriction of Δ

For θ compactly supported and $\Delta_\theta = \Delta|_{C_c^\infty(X) \cap \ker(\theta)}$,

- The Friedrichs extension $\widetilde{\Delta}_\theta$ is self-adjoint,
- the domain of $\widetilde{\Delta}_\theta$ lies inside $H^1(X)$, and
- for u in the domain,

$$(\widetilde{\Delta}_\theta - \lambda)u = 0 \quad \Longleftrightarrow \quad (\Delta - \lambda)u = (\text{const}) \cdot \theta$$

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$$(\widetilde{\Delta}_\theta - \lambda)u = 0 \quad \Longleftrightarrow \quad (\Delta - \lambda)u = (\text{const}) \cdot \theta$$

But: u will only be in $H^1(X)$ if θ is in $H^{-1}(X)$.

- $\theta = \delta_{z_0}^{\text{afc}}$ does not work: $\delta_{z_0}^{\text{afc}} \in H^\ell$ only for $\ell < -1$
- Try other choices of θ and domain of Δ_θ!

Project to Non-cuspidal Spectrum

Let Θ be a compactly supported distribution on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$.

$$\Theta \;=\; \sum_F \Theta(\overline{F})\cdot F + \Theta(\overline{\Phi_0})\cdot\Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{1-s})\cdot E_s\, ds$$

Following Colin de Verdiere (1983),

$$\theta \;=\; \mathrm{Proj}_{nc}\Theta \;=\; \Theta(\overline{\Phi_0})\cdot\Phi_0 + \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{1-s})\cdot E_s\, ds$$

Restrict Δ to $L^2_{nc}(X)\cap C^\infty_c(X)\cap \ker(\theta)$, and let $\widetilde{\Delta}_\theta$ be its Friedrichs extension. Then, for u in the domain of $\widetilde{\Delta}_\theta$,

$$(\widetilde{\Delta}_\theta - \lambda_w)u = 0 \quad\Longleftrightarrow\quad (\Delta-\lambda_w)u = (\mathrm{const})\cdot\theta$$

Theorem (Bombieri-Garrett)

Let $\theta = \text{Proj}_{nc}\Theta$, where Θ is a compactly supported distribution on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$. Let Δ_θ be the restriction of the Laplacian to $L^2(X)_{nc} \cap C_c^\infty(X) \cap \ker(\theta)$ and $\widetilde{\Delta}_\theta$ its Friedrichs extension.

Suppose θ lies in $H^{-1}(X)$ and θ is real, in the sense that $\theta(\overline{\varphi}) = \overline{\theta(\varphi)}$ for all $\varphi \in C_c^\infty(X)$

Then the compact period θE_w vanishes when $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\text{Re}(w) = \frac{1}{2}$.

Note

Hardy-Littlewood 1918 $\Rightarrow \theta = \text{Proj}_{nc}\delta_\omega^{afc}$ satisfies the hypotheses. In this case: $\theta E_s = \zeta_{\mathbb{Q}(\omega)}(s)/\zeta(2s)$.

Proof of Theorem

Let u be an eigenfunction for $\widetilde{\Delta}_\theta$ with eigenvalue $\lambda_w = w(w-1)$ and $w = \frac{1}{2} + i\tau$.

We aim to show that $\theta E_{\frac{1}{2}+i\tau} = 0$. Since the map $s \mapsto \theta E_s$ is continuous, it suffices to show

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\theta E_{\frac{1}{2}+it}|^2 \, dt \quad \to \quad 0 \qquad \text{as} \qquad \varepsilon \to 0$$

Since u lies in the domain of $\widetilde{\Delta}_\theta$, which is contained in $H^1(X)$, u has a spectral expansion converging in $H^1(X)$, thus also in $L^2(X)$. Plancherel ensures that the spectral coefficients are also square integrable. Let $A_s = A_{\frac{1}{2}+it}$ be the spectral coefficient corresponding to $E_s = E_{\frac{1}{2}+it}$.

On the other hand, we know that u is also a solution to $(\Delta - \lambda_w)u = \theta$, thus, by "engineering math"

$$A_{\frac{1}{2}+it} = \frac{\langle \theta, E_s \rangle}{\lambda_s - \lambda_w} = \frac{\overline{\theta(E_{\frac{1}{2}+it})}}{\tau^2 - t^2}$$

$$\Rightarrow \overline{\theta(E_{\frac{1}{2}+it})} = (\tau^2 - t^2) \cdot A_{\frac{1}{2}+it}$$

Since θ is real, and invoking Cauchy-Schwartz-Buniakovski,

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\theta E_{\frac{1}{2}+it}|^2 \, dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} |(\tau^2 - t^2) A_{\frac{1}{2}+it}|^2 \, dt$$

$$\leqslant \int_{\tau-\varepsilon}^{\tau+\varepsilon} |\tau^2 - t^2|^2 \, dt \int_{\tau-\varepsilon}^{\tau+\varepsilon} |A_{\frac{1}{2}+it}|^2 \, dt \ll \varepsilon^3 \cdot \|A_{\frac{1}{2}+it}\|^2_{L^2(\mathbb{R})}$$

Since $s \mapsto \theta E_s$ is continuous, this implies that $\theta E_w = 0$. □

Corollary

If $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\theta = \mathrm{Proj}_{\mathrm{nc}}\delta_\omega^{\mathrm{afc}}$, then $\zeta_{\mathbb{Q}(\omega)}(w)$ vanishes whenever w is on the critical line.

Corollary

If $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\theta = \mathrm{Proj}_{\mathrm{nc}}\delta_\omega^{\mathrm{afc}}$, then $\zeta_{\mathbb{Q}(\omega)}(w)$ vanishes whenever w is on the critical line.

Note

One would hope that the parameter set of eigenvalues would account for a large proportion of the zeros of the zeta function, thus proving that a large proportion of the zeros lie on the critical line. However, it turns out that we miss at least a positive fraction of zeros. In fact, it is not clear that the parameter set of eigenvalues is nonempty.

GL_3 Automorphic Spectral Theory

Consider $X = SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO(3)$.
$L^2(X) = L^2(X)_{csp} \oplus L^2(X)_{nc}$

- ONB of GL_3 spherical cusp forms $\{F\}$ for $L^2(X)_{csp}$
- Non-cuspidal spectrum:
  - Min. parabolic Eis. series $E_\chi^{1,1,1}$, $\chi \in \exp(\mu)$, $\mu \in \rho + i\mathfrak{a}^*$
  - $P^{2,1}$-Eis. series, $E_{f,s}^{2,1}$, f in onb of GL_2 cfms, $s \in \frac{1}{2} + i\mathbb{R}$
  - Constant afm Φ_0 with unit L^2-norm

For v in a GL_3 global automorphic Sobolev space,

$$v = \sum_{\text{cfm } F} \langle v, F\rangle \cdot F + \langle v, \Phi_0\rangle + \frac{1}{|W|}\int_{\rho+i\mathfrak{a}^*} \langle v, E_{\chi_\mu}^{1,1,1}\rangle \cdot E_{\chi_\mu}^{1,1,1}\, d\mu$$

$$+ \sum_{GL_2 \text{ cfms } f} \int_{\frac{1}{2}+i\mathbb{R}} \langle v, E_{f,s}^{2,1}\rangle \cdot E_{f,s}^{2,1}\, ds$$

Projecting to $L^2(X)_{nc,f}$

Let Θ be a compactly supported distribution on X.

$$\Theta \;=\; \sum_{\text{cfm } F} \Theta(\overline{F}) \cdot F \;+\; \frac{\Theta(1)}{\langle 1,1\rangle} \;+\; \frac{1}{|W|}\int_{\rho+i\mathfrak{a}^*} \Theta(E_{\overline{\chi}_\mu}) \cdot E_{\chi_\mu}\, d\mu$$

$$+ \sum_{GL_2 \text{ cfms } f} \int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{\overline{f},1-s}) \cdot E_{f,s}\, ds$$

Fix a GL_2 cusp form f, and project Θ:

$$\theta \;=\; \mathrm{Proj}_{nc,f}\,\Theta \;=\; \int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{\overline{f},1-s}) \cdot E_{f,s}\, ds$$

Restrict Δ to $L^2_{nc,f}(X) \cap C^\infty_c(X) \cap \ker(\theta)$, and let $\widetilde{\Delta}_\theta$ be its Friedrichs extension. Then, for u in the domain of $\widetilde{\Delta}_\theta$,

$$(\widetilde{\Delta}_\theta - \lambda)u \;=\; 0 \quad\Longleftrightarrow\quad (\Delta - \lambda)u \;=\; (\text{const}) \cdot \theta$$

Theorem

Let θ, Δ_θ, and $\widetilde{\Delta}_\theta$ be as above. If $\theta \in H^{-1}(X)$ and θ is real, then the compact period $\theta E_{f,w}$ vanishes when $\lambda_w = 2\lambda_f + 6w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$.

Note

Some compact periods of cuspidal data Eisenstein series turn out to be L-functions. The condition that θ lie in $H^{-1}(X)$ can be restated in terms of a second moment:

$$\int_0^T |\theta(E_{f,\frac{1}{2}+it})|^2 \, dt \quad \ll \quad T^{2-\epsilon_0} \qquad (\epsilon_0 > 0)$$

So, under subconvexity? Lindelöf? we can possibly prove that zeros of these L-functions corresponding to eigenvalues of $\widetilde{\Delta}_\theta$ lie on the critical line . . . hoping that there are “many” such . . .

Proof of Theorem

Let u be an eigenfunction for $\widetilde{\Delta}_\theta$ with eigenvalue $\lambda_w = 2\lambda_f + 6w(w-1)$, where $\lambda_f \in \mathbb{R}$ is the eigenvalue of the fixed GL_2 cusp form f and $w = \frac{1}{2} + i\tau$.

Since $u \in H^1(X)$, it has a spectral expansion with square integrable coefficients. Let $A_s = A_{\frac{1}{2}+it}$ be the spectral coefficient corresponding to $E_{f,s} = E_{f,\frac{1}{2}+it}$. Since u is also a solution to $(\Delta - \lambda_w)u = \theta$,

$$A_{\frac{1}{2}+it} \;=\; \frac{\langle \theta, E_{f,s}\rangle}{\lambda_{f,s} - \lambda_w} \;=\; \frac{\overline{\theta(E_{f,\frac{1}{2}+it})}}{6(\tau^2 - t^2)}$$

$$\Rightarrow \overline{\theta(E_{f,\frac{1}{2}+it})} \;=\; 6(\tau^2 - t^2)\cdot A_{\frac{1}{2}+it}$$

Proof, continued

Thus, invoking CSB as before,

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\theta E_{f,\frac{1}{2}+it}|^2 \, dt \;=\; 6\int_{\tau-\varepsilon}^{\tau+\varepsilon} |(\tau^2-t^2)A_{\frac{1}{2}+it}|^2 \, dt$$

$$\leqslant\; 6\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\tau^2-t^2|^2 \, dt \int_{\tau-\varepsilon}^{\tau+\varepsilon} |A_{\frac{1}{2}+it}|^2 \, dt \;\ll\; \varepsilon^3 \cdot \|A_{\frac{1}{2}+it}\|^2_{L^2(\mathbb{R})}$$

Since $s \mapsto \theta E_{f,s}$ is continuous, this implies that $\theta E_{f,w} = 0$. □

Thank you for your attention!