Zeros of Zeta Functions and Eigenvalues of Pseudo-Laplacians

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Abstract The occurrence of zeros of the Riemann zeta function in a list (Haas, 1977) of parameter values $\{s : \lambda_s = s(s-1)\}$ for purported eigenvalues λ_s of the Laplacian on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ raised hopes that a proof of the Riemann Hypothesis might be within reach, prompting a flurry of activity, trying first to reproduce, then correct or modify Haas' results. Although Hejhal showed that Haas' methods were flawed, the intriguing fact that his error would produce exactly the zeros of zeta led to related investigations (Hejhal, 1981, Colin de Verdière 1981 and 1983.) Recent work of Garrett and Bombieri sheds light on the previously hidden difficulties inherent this approach, opening the door to new constructions which avoid these difficulties. We will discuss some of these new constructions.

1 Introduction

1.1 Zeros of zeta functions and eigenvalues of self-adjoint operators

Our goal (following Hilbert and Polya) is to produce zeros of zeta functions (or other compact periods of automorphic forms) among parameters w for eigenvalues $\lambda_w = w(w-1)$ of self-adjoint operators.

We will prove a result that can be roughly stated as follows:

> If $\lambda_w = w(w-1)$ is an eigenvalue of a (carefully constructed) self-adjoint operator, "$\widetilde{\Delta}_\theta$", then the period θE_w vanishes when w is on the critical line.
>
> $$\{w \in \tfrac{1}{2} + i\mathbb{R} : \lambda_w = w(w-1) \text{ is an eigenvalue for } \widetilde{\Delta}_\theta\} \quad \subset \quad \{s : \theta E_s = 0\}$$

(Necessarily such an eigenvalue is real, so w lies either on the critical line or on the real interval $[0, 1]$.)

For example, we may want to choose $\theta = \delta_{z_0}^{\text{afc}}$, the Dirac delta distribution with base point z_0 chosen to be a CM point, e.g. $z = i$ or $z = \omega$, the corner of the fundamental domain. When $z_0 = \omega$, $\theta E_s = \zeta_{\mathbb{Q}(\omega)}(s)/\zeta(2s)$, and so vanishing of θE_s in the critical strip corresponds exactly to vanishing of $\zeta_{\mathbb{Q}(\omega)}$ in the critical strip. If we can construct a suitable operator "$\widetilde{\Delta}_\theta$" for $\theta = \delta_{z_0}^{\text{afc}}$ with non-empty (large!?) discrete spectrum, then we will be able to prove that a large subset of zeros of the Dedekind zeta function $\zeta_{\mathbb{Q}(\omega)}$ lie on the critical line!

Note. In retrospect, after Hejhal and Colin de Verdiere's work, we can view Haas' error as a failure to distinguish between Δ and this more subtle, still-to-be-described "$\widetilde{\Delta}_\theta$" with $\theta = \delta_\omega^{\text{afc}}$. This is why zeros of $\zeta_{\mathbb{Q}(\omega)}(s)$, including zeros of the Riemann zeta function (since $\zeta_{\mathbb{Q}(\omega)}$ factors as $\zeta(s)L(s, \chi)$, where χ is the non-trivial Dirichlet character mod 3), appeared among the parameter values for his purported eigenvalues of Δ.

1.2 Constructing suitable operators

The operator $\widetilde{\Delta}_\theta$ will be the Friedrichs extension (necessarily self-adjoint) of a suitable restriction (depending on θ) of the Laplacian Δ on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$.

Given a compactly supported automorphic distribution θ, we want to choose a suitable restriction Δ_θ of the Laplacian such that its Friedrichs extension $\widetilde{\Delta}_\theta$ satisfies:

$$(\widetilde{\Delta}_\theta - \lambda_w)u = 0 \quad \overset{??}{\Longleftrightarrow} \quad (\Delta - \lambda_w)u = (\text{const}) \cdot \theta$$

(We want the operator $\widetilde{\Delta}_\theta$ to be similar to Δ but to tolerate some non-smoothness in its eigenfunctions.) Since solutions to the latter equation can be found using "engineering math," this would give us a way to describe eigenfunctions for $\widetilde{\Delta}_\theta$!

However, it is possible that the "engineering math" solutions do not actually lie in the domain of $\widetilde{\Delta}_\theta$!

In particular, as Colin de Verdiere observed, it is possible to construct Δ_θ for $\theta = \delta_{z_0}^{\text{afc}}$ such that $\widetilde{\Delta}_\theta$ "overlooks" θ and to find explicit spectral expansions for solutions u by "engineering math", but it turns out that these solutions lie outside the domain of $\widetilde{\Delta}_\theta$, so they do not correspond to genuine eigenfunctions for $\widetilde{\Delta}_\theta$.

To understand this failure (and to make alternative constructions with more chance of success), it is necessary to understand some of the details of the Friedrichs extension as well as some global automorphic Sobolev theory.

2 Engineering Math, Sobolev Theory, and the Friedrichs Extension

2.1 Engineering math on $\mathbb{R}$: solving differential equations by division

To communicate the main ideas, we look at the simplest case: $\mathbb{R}$. Here $\Delta = \frac{d^2}{dx^2}$. Normalize λ_w as $\lambda_w = 4\pi^2 w^2$, and let $\theta = \delta$ be the Dirac delta distribution at $x = 0$ on $\mathbb{R}$. Our differential equation $(\Delta - \lambda_w)u = \theta$ becomes

$$\left(\tfrac{d^2}{dx^2} - 4\pi^2 w^2\right)u \;=\; \delta$$

Apply a Fourier transform to both sides:

$$(-4\pi^2\xi^2 - 4\pi^2 w^2)\mathcal{F}(u) \;=\; \mathcal{F}(\delta) = 1$$

This gives the Fourier coefficients for u, by division. The spectral expansion for u is given by Fourier inversion, and residue calculus gives an elementary result.

$$u(x) \;=\; \int_{\mathbb{R}} \mathcal{F}u(\xi)\, e^{2\pi i\xi x}\, d\xi \;=\; \frac{-1}{4\pi^2}\int_{\mathbb{R}} \frac{e^{2\pi i\xi x}}{\xi^2 + w^2}\, d\xi \;=\; \frac{-e^{2\pi w|x|}}{4\pi w} \qquad (\text{Re}(w) > 0)$$

More generally, if θ is a compactly supported distribution, the solution u to $\left(\frac{d^2}{dx^2} - 4\pi^2 w^2\right)u \;=\; \theta$ has spectral expansion

$$u \;=\; \frac{-1}{4\pi^2}\int_{\mathbb{R}} \frac{\theta(\overline{\psi}_\xi)\, e^{2\pi i\xi x}}{\xi^2 + w^2}\, d\xi \;=\; \int_{\mathbb{R}} \frac{\langle \theta, \psi_\xi\rangle \cdot \psi_\xi}{\lambda_\xi - \lambda_w} \qquad \text{where } \psi_\xi = e^{2\pi i x\xi},\;\; \lambda_\xi = -4\pi^2\xi^2$$

The issue of convergence is non-trivial, and Sobolev theory provides a robust framework for addressing it.

2.2 Justification using Sobolev spaces on $\mathbb{R}$

For positive integer ℓ, define an inner product $\langle\, ,\, \rangle_\ell$ on C_c^∞ by

$$\langle \varphi_1, \varphi_2\rangle_\ell = \langle (1-\Delta)^\ell \varphi_1, \varphi_2\rangle_{L^2} \qquad \text{where} \qquad \Delta = \Delta_{\mathbb{R}} = \tfrac{d^2}{dx^2}$$

Let H^ℓ be the Hilbert space completion of C_c^∞ with respect to the topology induced by $\langle\, ,\, \rangle_\ell$, and let $H^{-\ell}$ be its Hilbert space dual.

Note that $H^0 = L^2$ and the Sobolev spaces form a nested family, $H^\ell \hookrightarrow H^{\ell-1}$ for all ℓ.

The Laplacian acts nicely: $(1-\Delta) : H^\ell \to H^{\ell-2}$ is a Hilbert space isomorphism.

The spectral transform gives a Hilbert space isomorphism to a weighted L^2-space V^ℓ on the spectral side: $\mathcal{F} : H^\ell \to V^\ell$, and we have the following commutative diagram of Hilbert space isomorphisms:

$$\begin{array}{ccccccccccc}
\ldots & H^{+\ell} & \xrightarrow[\approx]{(1-\Delta)} & H^{+\ell-2} & \xrightarrow[\approx]{(1-\Delta)} & \ldots & \xrightarrow[\approx]{(1-\Delta)} & H^{-\ell+2} & \xrightarrow[\approx]{(1-\Delta)} & H^{-\ell} & \ldots \\
 & \mathcal{F}\big\downarrow\approx & & \mathcal{F}\big\downarrow\approx & & & & \mathcal{F}\big\downarrow\approx & & \mathcal{F}\big\downarrow\approx & \\
\ldots & V^{+\ell} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{+\ell-2} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & \ldots & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{-\ell+2} & \xrightarrow[\approx]{\times(1-\lambda_\xi)} & V^{-\ell} & \ldots
\end{array}$$

where, as above, $\lambda_\xi = -4\pi^2\xi^2$ is the Δ-eigenvalue of $\psi_\xi = e^{2\pi i x\xi}$.

This provides justification for our "engineering math." In particular, if θ is a distribution lying in some Sobolev space $H^{-\ell}$ (and all compactly supported distributions do, for some ℓ), then there is a solution u, unique in Sobolev spaces, satisfying $(\Delta - \lambda_w)u = \theta$. Further, u lies in $H^{-\ell+2}$, and has spectral expansion

$$u \;=\; \int_\Xi \frac{\langle \theta, \psi_\xi\rangle \cdot \psi_\xi}{\lambda_\xi - \lambda_w}\, d\xi \;=\; \frac{-1}{4\pi^2}\int_{\mathbb{R}} \frac{\theta(\overline{\psi}_\xi)\cdot e^{2\pi i \xi x}}{\xi^2 + w^2}\, d\xi$$

converging in $H^{-\ell+2}$.

The Sobolev embedding theorem allows comparison to C^k convergence: for $\ell > k + 1/2$, $H^\ell \hookrightarrow C^k$. For example, $H^1 \hookrightarrow C^0$, so if $\theta \in H^{-1}$, then the spectral expansion for u converges in H^1 and thus in C^0, i.e. uniformly pointwise. Note the shift-of-index.

2.3 Automorphic case

We have developed an analogous global Sobolev theory for the automorphic case, which interacts nicely with the spectral theory for automorphic forms.

Consider the case $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$. The analog of Fourier inversion is the automorphic spectral expansion in terms of eigenfunctions for the Laplacian $\Delta = y^2(\frac{d^2}{dx^2} + \frac{d^2}{dy^2})$.

$$v \;\stackrel{L^2}{=}\; \sum_F \langle v, F\rangle \cdot F \;+\; \langle v, \Phi_0\rangle \cdot \Phi_0 \;+\; \frac{1}{4\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \langle v, E_s\rangle \cdot E_s\, ds$$

where F ranges over an orthonormal basis of cusp forms, Φ_0 is the constant automorphic form with unit L^2-norm, and E_s is the real analytic Eisenstein series. (Note that the integrals (both the integral over s and the integrals implied by the pairings) are not necessarily uniformly pointwise convergent, but they can be understood as extensions by isometric isomorphisms of continuous linear functionals on $C_c^\infty(X)$.)

We can abbreviate (and generalize) this by denoting elements of the spectral "basis" (cusp forms, Eisenstein series, residues of Eisenstein series) uniformly as $\{\Phi_\xi\}_{\xi\in\Xi}$.

$$v \;=\; \int_\Xi^\oplus \langle v, \Phi_\xi\rangle \cdot \Phi_\xi \; d\xi$$

Global automorphic Sobolev theory allows us to conclude that:

- Every compactly supported distribution lies in $H^{-\ell}$ for some ℓ.

- Given $\lambda_w \in \mathbb{C}$ and a compactly supported θ, there is a unique u in Sobolev spaces such that $(\Delta - \lambda_w)u = \theta$. Further, if $\theta \in H^{-\ell}$, then $u \in H^{-\ell+2}$, and u has the following spectral expansion, converging in the $H^{-\ell+2}$-topology:

$$u \;=\; \int_{\Xi}^{\oplus} \frac{\langle \theta, \Phi_\xi \rangle \cdot \Phi_\xi}{\lambda_\xi - \lambda_w}\, d\xi$$

- A global automorphic Sobolev embedding theorem (for $\ell > k + \dim(X)/2$, $H^\ell \hookrightarrow C^k$) allows comparison to C^k convergence, and, in particular, uniform pointwise convergence, if desired.

Recall that we expect some correspondence between solutions of $(\Delta - \lambda_w)u = \theta$ and eigenfunctions for a suitably constructed "$\widetilde{\Delta}_\theta$". We now have a complete description of the solutions that lie in global automorphic Sobolev spaces. However, it is necessary to understand a few things about the details of the Friedrichs construction to see why it is that sometimes the solutions turn out *not* to be eigenfunctions.

2.4 The Friedrichs extension and the failure of $\theta = \delta_{z_0}^{\text{afc}}$

An *unbounded operator* on a Hilbert space is simply a linear map from a *subspace* (the domain) to the Hilbert space. In particular, specifying the domain is essential when defining such an operator.

The Laplacian Δ (restricted to a suitable dense subpace of $L^2(X)$, like $C_c^\infty(X)$) is a densely defined nonpositive symmetric unbounded operator on $L^2(X)$. Friedrichs' general construction gives a self-adjoint extension, which depends on the choice of domain.

For example, for θ a compactly supported automorphic distribution, let Δ_θ refer to the restriction of Δ to $C_c^\infty(X) \cap \ker(\theta)$. By construction, the Friedrichs extension $\widetilde{\Delta}_\theta$ is self-adjoint, its domain lies inside $H^1(X)$, and for u in the domain,

$$(\widetilde{\Delta}_\theta - \lambda)u = 0 \quad\iff\quad (\Delta - \lambda)u = (\text{const}) \cdot \theta$$

However, by the discussion above, the solution u will only be in $H^1(X)$ if θ is in $H^{-1}(X)$. This restricts our choices for θ significantly, and, in particular, this is why the choice $\theta = \delta_{z_0}^{\text{afc}}$ does not work: $\delta_{z_0}^{\text{afc}} \in H^\ell$ only for $\ell < -1$.

The framework of global automorphic Sobolev theory helps distinguish between aspects of the construction of $\widetilde{\Delta}_\theta$ that are inherent and aspects that can be modified. In particular, while $\theta = \delta_{z_0}^{\text{afc}}$ will certainly *not* work, other choices of θ and of the domain of Δ_θ may yield interesting results.

3 Projecting to the Non-cuspidal Spectrum: Positive Results

3.1 Projecting distributions to non-cuspidal spectrum for GL_2

Let Θ be a compactly supported distribution on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$. It has a spectral expansion converging in a negatively indexed Sobolev space:

$$\Theta \;=\; \sum_F \Theta(\overline{F}) \cdot F \;+\; \Theta(\overline{\Phi}_0) \cdot \Phi_0 \;+\; \frac{1}{4\pi i} \int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{1-s}) \cdot E_s \, ds$$

Following Colin de Verdiere (1983), we project to the non-cuspidal part of the spectrum, defining θ as

$$\theta \;=\; \text{Proj}_{\text{nc}}\Theta \;=\; \Theta(\overline{\Phi}_0) \cdot \Phi_0 \;+\; \frac{1}{4\pi i} \int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{1-s}) \cdot E_s \, ds$$

Consider Δ restricted to $L^2_{\text{nc}}(X) \cap C_c^\infty(X) \cap \ker(\theta)$, and let $\widetilde{\Delta}_\theta$ be its Friedrichs extension. Then, for u in the domain of $\widetilde{\Delta}_\theta$,

$$(\widetilde{\Delta}_\theta - \lambda_w)u = 0 \quad\iff\quad (\Delta - \lambda_w)u = (\text{const}) \cdot \theta$$

Thus λ_w is an eigenvalue for $\widetilde{\Delta}_\theta$ if an only if there is a nonzero solution u to the automorphic differential equation $(\Delta - \lambda_w)u = \theta$, lying in the domain of $\widetilde{\Delta}_\theta$.

Without the framework of global automorphic Sobolev spaces, Colin de Verdiere could only speculate the following result.

Theorem (Bombieri-Garrett)**.** Let $\theta = \mathrm{Proj}_{\mathrm{nc}}\Theta$, where Θ is a compactly supported distribution on $X = SL_2(\mathbb{Z})\backslash\mathfrak{H}$. Let Δ_θ be the restriction of the Laplacian to $L^2(X)_{\mathrm{nc}} \cap C_c^\infty(X) \cap \ker(\theta)$ and $\widetilde{\Delta}_\theta$ its Friedrichs extension. If θ lies in $H^{-1}(X)$ and θ is real, in the sense that $\theta(\overline{\varphi}) = \overline{\theta(\varphi)}$ for all $\varphi \in C_c^\infty(X)$, then the compact period θE_w vanishes when $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$.

Note. As Colin de Verdiere observed, the distribution $\theta = \mathrm{Proj}_{\mathrm{nc}}\delta_{z_0}^{\mathrm{afc}}$, with $z_0 = i, \omega$, satisfies the hypotheses in the theorem, by a 1970 result of Motohashi on the second moment of Dedekind zeta functions. (In fact, as Bombieri has pointed out, the 1918 result of Hardy and Littlewood suffices.) Recall that, in this case, with $z_0 = \omega$, the period θE_s is a ratio of zeta functions: $\theta E_s = \zeta_{\mathbb{Q}(\omega)}(s)/\zeta(2s)$.

Proof of Theorem. Let u be an eigenfunction for $\widetilde{\Delta}_\theta$ with eigenvalue $\lambda_w = w(w-1)$. Since $\widetilde{\Delta}_\theta$ is self-adjoint, by construction, w necessarily lies either on the critical line or on the real interval $[0,1]$. By hypothesis w lies on the critical line; let $w = \frac{1}{2} + i\tau$. We aim to show that $\theta E_{\frac{1}{2}+i\tau} = 0$. Since the map $s \mapsto \theta E_s$ is continuous, it suffices to show

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\theta E_{\frac{1}{2}+it}|^2 \, dt \quad \to \quad 0 \qquad \text{as} \qquad \varepsilon \to 0$$

Since u lies in the domain of $\widetilde{\Delta}_\theta$, which is contained in $H^1(X)$, u has a spectral expansion converging in $H^1(X)$, thus also in $L^2(X)$. Plancherel ensures that the spectral coefficients are also square integrable. Let $A_s = A_{\frac{1}{2}+it}$ be the spectral coefficient corresponding to $E_s = E_{\frac{1}{2}+it}$.

On the other hand, we know that u is also a solution to $(\Delta - \lambda_w)u = \theta$, thus, by "engineering math"

$$A_{\frac{1}{2}+it} \;=\; \frac{\langle \theta, E_s\rangle}{\lambda_s - \lambda_w} \;=\; \frac{\overline{\theta(E_{\frac{1}{2}+it})}}{\tau^2 - t^2} \qquad \text{i.e.} \qquad \overline{\theta(E_{\frac{1}{2}+it})} \;=\; (\tau^2 - t^2)\cdot A_{\frac{1}{2}+it}$$

Since θ is real, and invoking Cauchy-Schwartz-Buniakovski,

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\theta E_{\frac{1}{2}+it}|^2 \, dt \;=\; \int_{\tau-\varepsilon}^{\tau+\varepsilon} |(\tau^2 - t^2) A_{\frac{1}{2}+it}|^2 \, dt \;\leq\; \int_{\tau-\varepsilon}^{\tau+\varepsilon} |\tau^2 - t^2|^2 \, dt \int_{\tau-\varepsilon}^{\tau+\varepsilon} |A_{\frac{1}{2}+it}|^2 \, dt \;\ll\; \varepsilon^3 \cdot \|A_{\frac{1}{2}+it}\|^2_{L^2(\mathbb{R})}$$

Since $s \mapsto \theta E_s$ is continuous, this implies that $\theta E_w = 0$. □

Corollary. If $\lambda_w = w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\theta = \mathrm{Proj}_{\mathrm{nc}}\delta_\omega^{\mathrm{afc}}$, then $\zeta_{\mathbb{Q}(\omega)}(w)$ vanishes whenever w is on the critical line.

3.2 A variation for GL_3

Consider $X = SL_3(\mathbb{Z})\backslash\mathbb{H}^3$. As in the previous case $L^2(X)$ decomposes into a cuspidal part, for which we may choose an orthonormal basis of GL_3 spherical cusp forms $\{F\}$, and a non-cuspidal part. In the non-cuspidal spectrum are the continuous family of minimal parabolic Eisenstein series $E_\chi^{1,1,1}$ where $\chi = \exp(\mu)$, for some $\mu \in \rho + i\mathfrak{a}^*$, and the family of $P^{2,1}$-Eisenstein series, $E_{f,s}^{2,1}$, with cuspidal data f in an orthonormal basis of GL_2 cusp forms and complex parameter $s \in \frac{1}{2} + i\mathbb{R}$, along with the constant automorphic form Φ_0 (residue of minimal parabolic Eisenstein series). For v in a GL_3 global automorphic Sobolev space,

$$v \;=\; \sum_{\text{cfm } F} \langle v, F\rangle \cdot F \;+\; \langle v, \Phi_0\rangle \;+\; \frac{1}{|W|}\int_{\rho+i\mathfrak{a}^*} \langle v, E_{\chi_\mu}^{1,1,1}\rangle \cdot E_{\chi_\mu}^{1,1,1} \, d\mu \;+\; \sum_{GL_2 \text{ cfms } f} \int_{\frac{1}{2}+i\mathbb{R}} \langle v, E_{f,s}^{2,1}\rangle \cdot E_{f,s}^{2,1} \, ds$$

where convergence is in a global Sobolev topology. From now on, we drop the superscripts denoting the relevant parabolic for the Eisenstein series.

Let Δ the Laplacian on X and Θ a compactly supported distribution on X. Being compactly supported, Θ lies in a global automorphic Sobolev space and has the following spectral expansion:

$$\Theta \;=\; \sum_{\text{cfm } F} \Theta(\overline{F}) \cdot F \;+\; \frac{\Theta(1)}{\langle 1,1\rangle} \;+\; \frac{1}{|W|}\int_{\rho+i\mathfrak{a}^*} \Theta(E_{\overline{\chi}_\mu}) \cdot E_{\chi_\mu}\, d\mu \;+\; \sum_{GL_2 \text{ cfms } f} \int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{\overline{f},1-s}) \cdot E_{f,s}\, ds$$

Fix a GL_2 cusp form f, and project Θ to the corresponding part of the non-cuspidal spectrum: let

$$\theta \;=\; \mathrm{Proj}_{\mathrm{nc},f}\, \Theta \;=\; \int_{\frac{1}{2}+i\mathbb{R}} \Theta(E_{\overline{f},1-s}) \cdot E_{f,s}\, ds$$

Restrict Δ to $L^2_{\mathrm{nc},f}(X) \cap C_c^\infty(X) \cap \ker(\theta)$, and let $\widetilde{\Delta}_\theta$ be its Friedrichs extension. Then, for u in the domain of $\widetilde{\Delta}_\theta$,

$$(\widetilde{\Delta}_\theta - \lambda)u \;=\; 0 \qquad\Longleftrightarrow\qquad (\Delta - \lambda)u \;=\; (\text{const}) \cdot \theta$$

Theorem. Let θ, Δ_θ, and $\widetilde{\Delta}_\theta$ be as above. If θ lies in $H^{-1}(X)$ and θ is real, in the sense that $\theta(\overline{\varphi}) = \overline{\theta(\varphi)}$ for all $\varphi \in C_c^\infty(X)$, then the compact period $\theta E_{f,w}$ vanishes when $\lambda_w = 2\lambda_f + 6w(w-1)$ is an eigenvalue for $\widetilde{\Delta}_\theta$ with $\mathrm{Re}(w) = \frac{1}{2}$.

Note. Some compact periods of cuspidal data Eisenstein series turn out to be L-functions. The condition that θ lie in $H^{-1}(X)$ can be restated in terms of a second moment:

$$\int_0^T |\theta(E_{f,\frac{1}{2}+it})|^2\, dt \quad\ll\quad T^{2-\epsilon_0} \qquad\qquad (\epsilon_0 > 0)$$

So, under subconvexity? Lindelöf? we can possibly prove that zeros of these L-functions corresponding to eigenvalues of $\widetilde{\Delta}_\theta$ lie on the critical line ... hoping that there are "many" such ...

Proof of Theorem. Let u be an eigenfunction for $\widetilde{\Delta}_\theta$ with eigenvalue $\lambda_w = 2\lambda_f + 6w(w-1)$, where $\lambda_f \in \mathbb{R}$ is the eigenvalue of the fixed GL_2 cusp form f and $w = \frac{1}{2} + i\tau$. Since $u \in H^1(X)$, it has a spectral expansion with square integrable coefficients. Let $A_s = A_{\frac{1}{2}+it}$ be the spectral coefficient corresponding to $E_{f,s} = E_{f,\frac{1}{2}+it}$. Since u is also a solution to $(\Delta - \lambda_w)u = \theta$,

$$A_{\frac{1}{2}+it} \;=\; \frac{\langle \theta, E_{f,s}\rangle}{\lambda_{f,s} - \lambda_w} \;=\; \frac{\overline{\theta(E_{f,\frac{1}{2}+it})}}{6(\tau^2 - t^2)} \qquad \text{i.e.} \qquad \overline{\theta(E_{f,\frac{1}{2}+it})} \;=\; 6(\tau^2 - t^2) \cdot A_{\frac{1}{2}+it}$$

Thus, invoking CSB as before,

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\theta E_{f,\frac{1}{2}+it}|^2\, dt \;=\; 6\int_{\tau-\varepsilon}^{\tau+\varepsilon} |(\tau^2-t^2) A_{\frac{1}{2}+it}|^2\, dt \;\le\; 6\int_{\tau-\varepsilon}^{\tau+\varepsilon} |\tau^2-t^2|^2\, dt \int_{\tau-\varepsilon}^{\tau+\varepsilon} |A_{\frac{1}{2}+it}|^2\, dt \;\ll\; \varepsilon^3 \cdot \|A_{\frac{1}{2}+it}\|^2_{L^2(\mathbb{R})}$$

Since $s \mapsto \theta E_{f,s}$ is continuous, this implies that $\theta E_{f,w} = 0$. $\square$