Explicit fundamental solution for $(\Delta - \lambda)^\nu$ on G/K

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1. Introduction

With an eye towards certain number theoretic applications, we are interested in finding an **explicit** expression for the solution of this differential equation:

$$(\Delta - \lambda)^\nu \, u \;\; = \;\; \delta \qquad \text{on } \; G/K$$

where G is a complex semi-simple Lie group, K its maximal compact, Δ is the Casimir operator descended to right K-invariant functions (i.e. the Laplacian), λ is a complex (eigenvalue) parameter, ν is an integer, and δ is the Dirac delta distribution at the basepoint $x_o = 1 \cdot K$.

Remarks:

- u is a **smoothed** version of Dirac δ.
- δ is **bi-K-invariant**, so we can use the harmonic analysis of bi-K-invariant functions
- global zonal spherical Sobolev theory $\Rightarrow$ sufficiently high ν guarantees **continuity** of u

Remark: We are not just interested in the **existence** of a fundamental solution (to prove solvability of a differential operator), but in the **explicit expression** for it, with an eye towards number theoretic applications involving the associated Poincaré series.

2. Harmonic analysis of spherical functions

The spherical transform of Harish-Chandra, Berezin:

$$\mathcal{F}f(\xi) \;=\; \int_G f(g)\,\overline{\varphi}_{\rho+i\xi}(g)\,dg$$

where $\varphi_{\rho+i\xi}$ is the zonal spherical function (associated to the principal series $I_{\rho+i\xi}$). The inverse transform:

$$\mathcal{F}^{-1}f(g) \;=\; \int_{\mathfrak{a}^*} f(\xi)\,\varphi_{\rho+i\xi}(g)\,|\mathbf{c}(\xi)|^{-2}\,d\xi$$

where $\mathbf{c}$ is the Harish-Chandra $\mathbf{c}$-function.

Since δ is a compactly supported distribution, it lies in a negatively indexed global zonal spherical Sobolev space. The spherical transform extends to negatively indexed Sobolev spaces:

$$\begin{array}{ccccccccc}
\cdots & H^{+\ell} & \xrightarrow{(1-\Delta)} & H^{+\ell-2} & \xrightarrow{(1-\Delta)} & \cdots & \xrightarrow{(1-\Delta)} & H^{-k} & \cdots \\
 & \mathcal{F}\big\downarrow\approx & & \mathcal{F}\big\downarrow\approx & & & & \mathcal{F}\big\downarrow\approx & \\
\cdots & V^{+\ell} & \xrightarrow{\times(1-\lambda_\xi)} & V^{+\ell-2} & \xrightarrow{\times(1-\lambda_\xi)} & \cdots & \xrightarrow{\times(1-\lambda_\xi)} & V^{-k} & \cdots
\end{array}$$

where $\lambda_\xi \;=\; -(|\xi|^2+|\rho|^2)$ is the Casimir eigenvalue of $\varphi_{\rho+i\xi}$. Thus if u_z is the solution to $(\Delta-\lambda_z)^\nu\,u_z=\delta$ with $\lambda_z=z^2-|\rho|^2$,

$$u_z \;=\; \mathcal{F}^{-1}\!\left(\frac{\mathcal{F}(\delta)}{(\lambda_\xi-\lambda_z)^\nu}\right) \;=\; \mathcal{F}^{-1}\!\left(\frac{(-1)^\nu}{(|\xi|^2+z^2)^\nu}\right)$$

This gives us an **integral representation** for u. When G is complex, spherical functions are **elementary**, and we can evaluate the integral to obtain an explicit expression for u.

3. Integral representation for fundamental solution

For G a **complex**, semi-simple Lie group, spherical functions are elementary:

$$\varphi_{\rho+i\xi}(k \cdot a \cdot k') \;=\; \frac{\pi^+(\rho)}{\pi^+(i\xi)} \; \frac{\sum \operatorname{sgn} w \; e^{i\, w\xi(\log a)}}{\sum \operatorname{sgn} w \; e^{w\rho(\log a)}}$$

where the sums are over elements w in the Weyl group, ρ is half the sum of positive roots, and the function π^+ on $\mathfrak{a}^*$ is

$$\pi^+(\mu) \;=\; \prod_{\alpha>0} \langle \alpha, \mu \rangle$$

This is a homogeneous, degree d ($=$ the number of positive roots) polynomial, W-equivariant by the sign function. The ratio of $\pi^+(\rho)$ to $\pi^+(i\xi)$ the Harish-Chandra $\mathbf{c}$-function.

Thus the inverse spherical transform is:

$$\mathcal{F}^{-1} f \;=\; \frac{(-i)^d \, |W|}{\pi^+(\rho) \; \sum \operatorname{sgn} w \; e^{w\rho}} \int_{\mathfrak{a}^*} f(\xi) \; \pi^+(\xi) \; e^{i\xi} \; d\xi$$

(Looks like the Euclidean inverse Fourier transform of $f \cdot \pi^+$!) In particular,

$$u_z \;=\; \frac{(-i)^d \, |W|}{\pi^+(\rho) \; \sum \operatorname{sgn} w \; e^{w\rho}} \int_{\mathfrak{a}^*} \frac{(-1)^\nu}{(|\xi|^2 + z^2)^\nu} \; \pi^+(\xi) \; e^{i\xi} \; d\xi$$

4. Evaluating the integral

We wish to evaluate the integral

$$I(\log a) \;=\; \int_{\mathfrak{a}^*} \frac{(-1)^\nu}{(|\xi|^2 + z^2)^\nu} \; \pi^+(\xi) \; e^{i\langle \xi, \log a \rangle} \; d\xi$$

The crucial point is that π^+ is a **harmonic** homogeneous polynomial (with respect to the Euclidean Laplacian on $\mathfrak{a}^*$.) This allows us to use

Hecke's identity, which gives a simple rule for evaluating the Fourier transform of the product of a gaussian with a harmonic homogeneous polynomial:

$$\mathrm{FT}\big(e^{-|x|^2} \cdot P_d(x)\big) \;=\; (-i)^d \cdot e^{-|\xi|^2} \cdot P_d(\xi)$$

Using an identity with the Gamma function and changing variables,

$$I(\log a) \;=\; \frac{1}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-d/2-n/2}\, e^{-tz^2} \int_{\mathfrak{a}^*} e^{-|\xi|^2}\, \pi^+(\xi)\, e^{-i\langle \xi, -(\log a)/\sqrt{t}\rangle}\, d\xi\, \frac{dt}{t}$$

Applying Hecke's identity to the inner integral yields,

$$\int_{\mathfrak{a}^*} e^{-|\xi|^2}\, \pi^+(\xi)\, e^{-i\langle \xi, -(\log a)/\sqrt{t}\rangle}\, d\xi \;=\; (-i)^d e^{-|\log a|^2/t}\, \pi^+(-(\log a)/\sqrt{t})$$

Thus,

$$I(\log a) \;=\; \frac{i^d \cdot \pi^+(\log a)}{\Gamma(\nu)} \cdot \int_0^\infty t^{\nu-d-n/2}\; e^{-tz^2}\; e^{-|\log a|^2/t}\; \frac{dt}{t}$$

Inserting the Fourier transform of the gaussian and using the Γ-function identity again yields a rotation-invariant integral:

$$I(\log a) \;=\; \frac{i^d\, \Gamma(\nu - d)}{\Gamma(\nu)} \cdot \pi^+(\log a) \cdot \int_{\mathfrak{a}^*} \frac{e^{i\langle \xi, \log a\rangle}}{(|\xi|^2 + z^2)^\nu}\, d\xi$$

The rotation-invariance, allows us to assume $\langle \xi, \log a\rangle = \xi_1 \cdot |\log a|$. Using the Γ-function identity again and integrating a gaussian over $\mathbb{R}^{n-1}$ yields an integral over $\mathbb{R}$:

$$I(\log a) \;=\; \frac{i^d\, \pi^{(n-1)/2}\Gamma(\nu - d - \frac{n-1}{2})}{\Gamma(\nu)} \cdot \pi^+(\log a) \cdot \int_{\mathbb{R}} \frac{e^{i\xi_1 \cdot |\log a|}}{(|\xi_1|^2 + z^2)^{\nu - d - (n-1)/2}}\, d\xi_1$$

For n odd, let $\nu = d + (n+1)/2$, for continuity. Then the integral may be evaluated by residues.

$$I(\log a) \;=\; \frac{i^d\, \pi^{(n+1)/2}}{\Gamma(d + (n+1)/2)} \;\cdot\; \pi^+(\log a) \;\cdot\; \frac{e^{-z|\log a|}}{z}$$

5. Explicit expression for fundamental solution

Thus the fundamental solution (for odd rank) is:

$$\begin{aligned} u_z(k \cdot a \cdot k') &= \frac{|W|\pi^{(n+1)/2}}{\pi^+(\rho)\,\Gamma(d+(n+1)/2)} \cdot \frac{\pi^+(\log a)}{\sum \operatorname{sgn} w\; e^{w\rho}} \cdot \frac{e^{-z|\log a|}}{z} \\ &= C_G \cdot \prod_{\alpha>0} \frac{\alpha(\log a)}{2\sinh(\alpha(\log a)/2)} \cdot \frac{e^{-z|\log a|}}{z} \end{aligned}$$

Remarks:

- For n **even**, choosing ν for continuity results in a K-Bessel function.

- For $SL_2(\mathbb{C})$, the fundamental solution is

  $$u_z(r) = \frac{\pi\; r\; e^{-(2z-1)r}}{(2z-1)\sinh(r)}$$

  where r is the Cartan radius.

- For higher rank, $\log a$ is **not** merely a “Cartan radius,” since an element of G has “length” in several directions. It is perhaps more accurate to call it a “multi-radius.” Then $|\log a|$ is the “root mean square” of the multi-radius.

- The function

  $$\prod \frac{\alpha}{2\sinh(\alpha/2)}$$

  has **removable singularities** along hyperplanes perpendicular to the roots. It has **exponential decay** in all directions, but of varying orders, the greatest decay being along hyperplanes perpendicular to the roots.

6. The homogenous polynomial π^+ is harmonic

Recall that the crucial fact for evaluating the integral for u_z was the **harmonic**-ness of the function π^+ with respect to the Euclidean Laplacian on $\mathfrak{a}^*$. We will sketch the proof of this fact.

Since $\mathfrak{a}^*$ is Euclidean, its Lie algebra can be identified with itself. For any basis $\{x_i\}$ of the Lie algebra, the Casimir operator (Laplacian) is

$$\Delta \;=\; \sum_i x_i\, x_i^*$$

Consider as a warm-up the product of two linear functionals:

$$\Delta\,\langle\alpha,-\rangle\langle\beta,-\rangle \;=\; \sum_i x_i\, x_i^* \,\big(\langle\alpha,-\rangle\langle\beta,-\rangle\big)$$

$$=\; \sum_i x_i \,\big(\langle\alpha,x_i^*\rangle\langle\beta,-\rangle \;+\; \langle\alpha,-\rangle\langle\beta,x_i^*\rangle\big)$$

$$=\; \sum_i \big(\langle\alpha,x_i^*\rangle\langle\beta,x_i\rangle \;+\; \langle\alpha,x_i\rangle\langle\beta,x_i^*\rangle\big) \;=\; 2\langle\alpha,\beta\rangle$$

Now, applying Δ to the product over all positive roots,

$$\Delta\,\pi^+ \;=\; \sum_i x_i\, x_i^*\,\pi^+ \;=\; \sum_i x_i \;\sum_{\alpha>0}\langle\alpha,x_i^*\rangle\cdot\frac{\pi^+}{\alpha}$$

$$=\; \sum_i\;\sum_{\alpha>0}\langle\alpha,x_i^*\rangle\cdot x_i\;\frac{\pi^+}{\alpha} \;=\; \sum_i\;\sum_{\alpha>0}\langle\alpha,x_i^*\rangle\cdot\sum_{\beta\neq\alpha}\langle\beta,x_i\rangle\cdot\frac{\pi^+}{\alpha\beta}$$

$$=\; \sum_{\alpha\neq\beta}\;\sum_i\langle\alpha,x_i^*\rangle\;\langle\beta,x_i\rangle\cdot\frac{\pi^+}{\alpha\beta} \;=\; \sum_{\alpha\neq\beta} 2\langle\alpha,\beta\rangle\cdot\frac{\pi^+}{\alpha\beta}$$

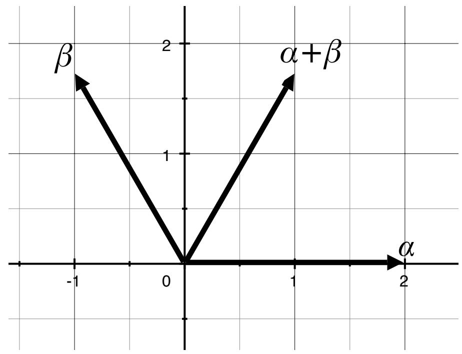
Thus (Lemma) π^+ is harmonic if

$$\sum_{\alpha\neq\beta}\frac{\langle\alpha,\beta\rangle}{\alpha\beta} \;\neq\; 0$$

where the sum runs over pairs of distinct positive roots.

We prove the Lemma for the simple rank two Lie algebras $\mathfrak{sl}_3$, $\mathfrak{sp}_2$, $\mathfrak{g}_2$.

For $\mathfrak{sl}_3$, the positive roots are α, β, and $\alpha+\beta$:



By the Lemma, it suffices to show

$$0 \;=\; \frac{\langle\alpha,\beta\rangle}{\alpha\,\beta} + \frac{\langle\alpha,\alpha+\beta\rangle}{\alpha(\alpha+\beta)} + \frac{\langle\beta,\alpha+\beta\rangle}{\beta(\alpha+\beta)}$$

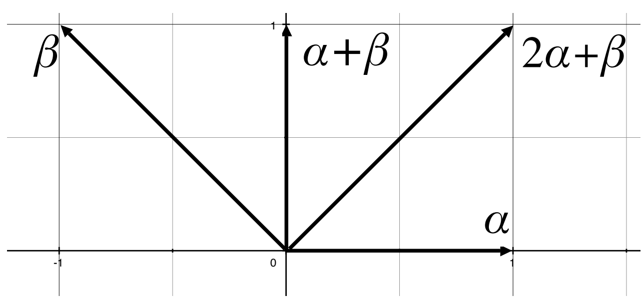
i.e. it suffices to show

$$0 \;=\; \langle\alpha,\beta\rangle\,(\alpha+\beta) \;+\; \langle\alpha,\alpha+\beta\rangle\,\beta \;+\; \langle\beta,\alpha+\beta\rangle\,\alpha$$

but this is

$$(-1)(\alpha+\beta) \;+\; \beta \;+\; \alpha \;=\; 0$$

For $\mathfrak{sp}_2$, the positive roots are α, β, $\alpha+\beta$, $2\alpha+\beta$:



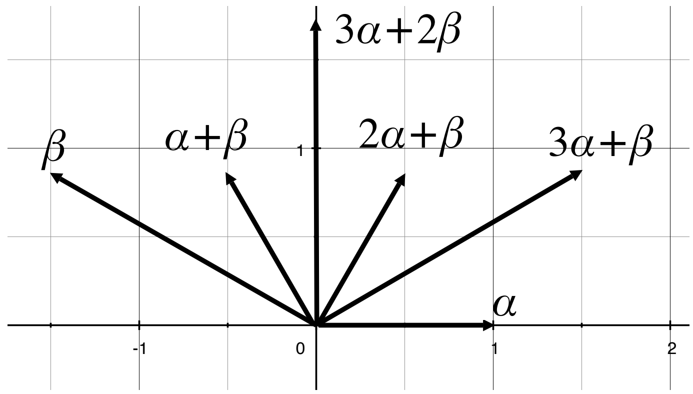
By the Lemma (and clearing denominators) it suffices to show:

$$0 \;=\; \langle\alpha,\beta\rangle\,(\alpha+\beta)(2\alpha+\beta) \;+\; \langle\alpha,2\alpha+\beta\rangle\,\beta(\alpha+\beta)$$
$$+\,\langle 2\alpha+\beta,\alpha+\beta\rangle\,\alpha\beta \;+\; \langle\alpha+\beta,\beta\rangle\,\alpha(2\alpha+\beta)$$

But this is

$$(-1)(2\alpha^2+3\alpha\beta+\beta^2) \;+\; \alpha\beta \;+\; \beta^2 \;+\; \alpha\beta \;+\; 2\alpha^2 \;+\; \alpha\beta \;=\; 0$$

For $\mathfrak{g}_2$, the positive roots are α, β, $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, $3\alpha+2\beta$:



Notice that α, $\alpha+\beta$, and $2\alpha+\beta$ form a copy of the $\mathfrak{sl}_3$ root system. Thus the three terms corresponding to the three pairs among these roots sum to zero. Similarly the three terms corresponding to the three pairs among $3\alpha+\beta$, β, and $3\alpha+2\beta$ sum to zero. There are six pairs of distinct positive roots left. They also sum to zero, but I will spare you the computation.

Thus we have shown that π^+ is harmonic when the group is of rank two.

The arbitrary case reduces to the rank two case. We need to show that

$$\sum_{\alpha\neq\beta} \frac{\langle\alpha,\beta\rangle}{\alpha\beta} \;=\; 0$$

Key Idea: divide up the pairs of roots into mutually disjoint sets corresponding to 2-dimensional root systems.

Each (nonzero) term corresponds to a pair (α, β) of disjoint (nonorthogonal) roots. Let $\mathcal{R}_{\alpha,\beta}$ be the 2-dimensional root system generated by α and β. For such a root system $\mathcal{R}$, let $I_{\mathcal{R}}$ be the set of pairs of roots in $\mathcal{R}$. The union of all such $I_{\mathcal{R}}$ is the indexing set for the sum. We will refine $\{I_{\mathcal{R}}\}$ to a cover of mutually disjoint sets.

If $I_{\mathcal{R}}$ intersects $I_{\mathcal{R}'}$ nontrivially, then the 2-dimensional root systems $\mathcal{R}$ and $\mathcal{R}'$ share a **pair** of roots, so lie in the **same plane**. Thus there must be a 2-dimensional root system $\mathcal{R}''$ containing both $\mathcal{R}$ and $\mathcal{R}'$.

So we refine the cover by replacing $I_{\mathcal{R}}$ and $I_{\mathcal{R}'}$ by $I_{\mathcal{R}''}$. Then, grouping the terms in the sum according to the sets in the refined cover, each group of terms sums to zero, since it corresponds to one of the rank two cases discussed above, and thus the entire sum is zero.