Harmonic Analysis of GL_2 and GL_3 Automorphic Forms

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The study of spectral theory of automorphic forms began with Rankin and Selberg in the late 1930's, continued with Selberg and Roelcke in the 50's, Gelfand, Fomin, and Graev in the 50's and 60's, Harish-Chandra and Langlands in the 60's, and more recently Moeglin and Walsdpurger in the 90's.

1. Automorphic Forms on $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$

First we treat the simplest possible case, automorphic forms on $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$. This corresponds to the familiar discussion of $SL_2(\mathbb{Z})$-invariant functions on the upper half plane. From a modern point of view, considering $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ corresponds to picking the archimedean place out of the more natural and coherent adelic version of the story, the harmonic analysis of automorphic forms on $Z_{\mathbb{A}}GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$. However, many of the difficulties that arise in the adelic discussion are already present in the archimedean case, so we choose to treat the archimedean case first.

Here $G = SL_2$. Recall the Iwasawa decomposition:

$$G = PK = MNK$$

where P is the standard parabolic subgroup (upper triangular matrices), K is the standard maximal compact subgroup (the orthogonal group $SO(2)$), M is the standard Levi component of P (diagonal matrices), and N is the unipotent radical for P (upper triangular matrices with 1's on the diagonal.)

Reduction theory and the theory of compact operators show that the space $L^2_{\text{cusp}}(G_{\mathbb{Z}}\backslash G_{\mathbb{R}})$ of square-integrable cusp forms decomposes discretely with finite multiplicity, i.e. for f in $L^2(G_{\mathbb{Z}}\backslash G_{\mathbb{R}})$ satisfying the Gelfand condition:

$$f \stackrel{L^2}{=} \sum_{F\in \text{ onb of cfms}} \langle f, F\rangle \cdot F$$

The orthogonal complement is spanned by *pseudo-Eisenstein series*

$$\Psi_\varphi(g) = \sum_{\gamma\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}} \varphi(\gamma g) \quad \text{for } \varphi \in C^0_c(N_{\mathbb{R}}M_{\mathbb{Z}}\backslash G_{\mathbb{R}})$$

The space of pseudo-Eisenstein series decomposes as the direct integral of Eisenstein series E_s, parameterized by $s \in \mathbb{C}$,

$$E_s(g) = \sum_{\gamma\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}} f_s(\gamma g)$$

where f_s is a vector in an induced representation, coming from a character χ_s on M extended to P by left N-invariance. This is the same as the classical presentation because

$$N_{\mathbb{R}}M_{\mathbb{Z}}\backslash G_{\mathbb{R}}/K \approx M_{\mathbb{Z}}\backslash M_{\mathbb{R}}$$

and furthermore:

$$Z_{\mathbb{R}}M_{\mathbb{Z}}\backslash M_{\mathbb{R}} \approx GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R}) \approx Z^\times\backslash\mathbb{R}^\times \approx \mathbb{R}^+$$

So if we assume that everything is *spherical*, with trivial central character, we can consider φ and χ_s as functions on $\mathbb{R}^+$. So we can use Mellin inversion to decompose φ:

$$\varphi = \int \mathcal{M}\varphi(s)\, y^s\, ds = \int \langle \varphi, \chi_s\rangle \cdot \chi_s$$

and so, after some work, the pseudo-Eisenstein series decomposes as:

$$\Psi_\varphi = \int \langle \Psi_\varphi, E_s\rangle \cdot E_s$$

Note that we need $\mathrm{Re}(s) > 1$ in order to do the manipulations required to obtain this decomposition. To fold up the integral, we move the contour to the critical line, picking up a residue. After some work,

$$\Psi_\varphi = \int_{\frac{1}{2}+i\mathbb{R}^+} \langle \Psi_\varphi, E_s \rangle \cdot E_s \quad + \quad \frac{\langle \Psi_\varphi, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$$

2. Harmonic Analysis for Automorphic Forms on $Z_\mathbb{A} GL_3(k) \backslash GL_3(\mathbb{A})$

Having discussed the spectral theory for $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ we now use the same framework to discuss the GL_3 case. Here we work in an adelic setting, over an arbitrary number field k. As we will see, this only alters the discussion minimally.

2.1 Classifying/grouping Automorphic Forms by Cuspidal Support

The goal is to decompose the space of square-integrable automorphic forms into irreducible subrepresentations. As in the case of SL_2, we start with the cuspidal automorphic forms.

Given a parabolic P in G, and function f on $Z_\mathbb{A} G_k \backslash G_\mathbb{A}$, the constant term of f along P is

$$c_P f(g) = \int_{N_k \backslash N_\mathbb{A}} f(ng)\, dn$$

where N is the unipotent radical of P. An automorphic form satisfies the Gelfand condition if, for all maximal parabolics P, the constant term along P is zero. If such a function is also $\mathfrak{z}$-finite (for example, it is an eigenfunction for Casimir) and K-finite (for example, it is spherical), it is called a cusp form.

Since the right action of G commutes with taking constant terms, the space of functions satisfying the Gelfand condition is G-stable, and so is a subrepresentation. Gelfand and Pietesky-Shapiro showed that integral operators on this space are compact, so by spectral theory of compact operators, this subrepresentation decomposes into a direct sum of irreducibles, each with finite multiplicity. We will take this for granted and decompose the rest of L^2.

Having used the constant term to filter out the cusp forms, we now use the map for further sorting. As a first step towards obtaining the L^2 decomposition of the non-cuspidal automorphic forms, we classify them according to their *cuspidal support*, i.e. the smallest parabolic on which they have a non-zero constant term. (Conversely we can think of the largest parabolic on which its constant term is zero.)

In GL_3, there are three conjugacy classes of proper parabolic subgroups. In addition, the whole group may also considered to be parabolic in a trivial sense. We will consider the standard parabolic subgroups: $P^3 = GL_3$, $P^{2,1}$ and $P^{1,2}$ the maximal parabolics, and $P^{1,1,1}$ the minimal parabolic, contained in both $P^{2,1}$ and $P^{1,2}$.

Starting with the easiest cases, we observe that an automorphic form whose constant term along $P^3 = GL_3$ is zero is identically zero, and an automorphic form with cuspidal support P^3 is precisely a nonzero cusp form.

There is more to say about automorphic forms whose cuspidal support is a maximal parabolic. Consider an automorphic form f with cuspidal support $P^{2,1}$ and let $F = c_{2,1} f$. Then F is a non-zero left $N^{2,1}$-invariant function. So if it is spherical, it can be considered as a GL_2 automorphic form. In fact it is a GL_2 cusp form, since the constant term of f along the minimal parabolic is zero.

Lastly, we have the automorphic forms whose cuspidal support is the minimal parabolic, i.e. those whose constant term along $P^{1,1,1}$ is nonzero.

While classifying automorphic forms according to cuspidal support is helpful (because of certain adjointness relations, which allow us to prove the orthogonality of subspaces spanned by them) it does not give us a very concrete or explicit description of the various classes of automorphic forms. Recall from the SL_2 case that pseudo-Eisenstein series provided an explicit description of automorphic forms with cuspidal support P, and the space spanned by pseudo-Eisenstein series was the orthogonal complement to the space of cusp forms. In GL_3 things are more complicated, since there are more parabolic subgroups, but we will still use pseudo-Eisenstein series to describe the orthogonal complement to the space of cusp forms.

Define pseudo-Eisenstein series in a manner exactly analogous to the SL_2 case:

$$\Psi_\varphi(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma \cdot g)$$

where φ is a continuous, compactly supported function on $Z_{\mathbb{A}} N_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}$. In GL_3, there are three different kinds of pseudo-Eisenstein series, corresponding to the three standard parabolic subgroups. It is relatively straightforward to check that the space of all pseudo-Eisenstein series is the orthogonal complement to the space of cusp forms, but it will require more work to determine how many different kinds of pseudo-Eisenstein series we actually need in order to span the complement.

We start with the following adjointness relation, the key to proving orthogonality.

Claim. *For any square-integrable automorphic form f, and any pseudo-Eisenstein series Ψ_φ^P, with P a parabolic subgroup,*

$$\langle f, \Psi_\varphi^P \rangle_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} = \langle c_P f, \varphi \rangle_{Z_{\mathbb{A}} N_{\mathbb{A}}^P M_k^P \backslash G_{\mathbb{A}}}$$

Proof. This is a standard winding/unwinding argument:

$$\begin{aligned}
\langle f, \Psi_\varphi^P \rangle_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} &= \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} f(g) \overline{\Psi_\varphi^P(g)} \, dg \\
&= \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} f(g) \cdot \Big(\sum_{\gamma \in P_k \backslash G_k} \overline{\varphi(\gamma g)} \Big) \, dg \\
&= \int_{Z_{\mathbb{A}} P_k \backslash G_{\mathbb{A}}} f(g) \overline{\varphi(g)} \, dg \\
&= \int_{Z_{\mathbb{A}} N_k M_k \backslash G_{\mathbb{A}}} f(g) \overline{\varphi(g)} \, dg \\
&= \int_{Z_{\mathbb{A}} N_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} f(ng) \overline{\varphi(ng)} \, dn \, dg \\
&= \int_{Z_{\mathbb{A}} N_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}} \Big(\int_{N_k \backslash N_{\mathbb{A}}} f(ng) \, dn \Big) \overline{\varphi(g)} \, dg \\
&= \langle c_P f, \varphi \rangle_{Z_{\mathbb{A}} N_{\mathbb{A}}^P M_k^P \backslash G_{\mathbb{A}}}
\end{aligned}$$

□

Note. This "winding/unwinding" is a specific example of integrating over quotients. For a G a topological group and H a closed subgroup,

$$\int_G f(g) \, dg = \int_{H \backslash G} \int_H f(hg) \, dh \, dg$$

as long as the modular functions are compatible. When the subgroup is discrete, we write the integral over the subgroup as a sum.

From this adjointness relation, it quickly follows that a (square-integrable) automorphic form is a cusp form if and only if it is orthogonal to all pseudo-Eisenstein series, i.e. the orthogonal complement to the space of cusp forms is spanned by pseudo-Eisenstein series.

Further, we can use this adjointness relation to decompose the space spanned by pseudo-Eisenstein series into orthogonal subspaces. In particular, if f is in the space spanned by pseudo-Eisenstein series, then it quickly follows from the adjointness relation that f has cuspidal support $P^{2,1}$ or $P^{1,2}$ if and only if it is orthogonal to all $P^{1,1,1}$ pseudo-Eisenstein series. So the orthogonal complement to cusp forms decomposes into two orthogonal subspaces: the space spanned by $P^{1,1,1}$ pseudo-Eisenstein series, and the space of automorphic forms with cuspidal support $P^{2,1}$ or $P^{1,2}$.

We need to determine which pseudo-Eisenstein series are in the second subspace. Although we might naively guess that all $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series have cuspidal support $P^{2,1}$ or $P^{1,2}$, this is not the case. What *is* true is that a $P^{2,1}$ or $P^{1,2}$ pseudo-Eisenstein series with cuspidal data (i.e. one whose data can be identified with a GL_2 cusp form) has cuspidal support $P^{2,1}$ or $P^{1,2}$. (To show this we need to compute the constant term along $P^{1,1,1}$ of such a pseudo-Eisenstein series. This computation is not trivial to carry out, and it relies on the Bruhat decomposition of GL_3. See the appendix on constant terms for further details.) Any other $P^{2,1}$ or $P^{1,2}$ pseudo-Eisenstein series (i.e. one with non-cuspidal data) can be written as the sum of a $P^{1,1,1}$ pseudo-Eisenstein series and a $P^{2,1}$ or $P^{1,2}$ pseudo-Eisenstein series with cuspidal data. So the subspace consisting of automorphic forms with cuspidal support $P^{2,1}$ or $P^{1,2}$ is spanned by $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series with cuspidal data.

As we will see, the space generated by $P^{1,2}$ pseudo-Eisenstein series is actually the *same* as the space generated by $P^{2,1}$ pseudo-Eisenstein series. This is an example of a more general phenomenon: pseudo-Eisenstein series of associate parabolics span the same space.

So we have the following decomposition of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ into orthogonal subspaces:

$$L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}) = (\text{cfms}) \oplus (\text{span of } P^{1,1,1} \text{ ps-Eis}) \oplus (\text{span of } P^{2,1} \text{ ps-Eis, cspdl data})$$

2.2 Decomposing Pseudo-Eisenstein Series

While we have a fairly nice description of the non-cuspidal automorphic forms in $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ in terms of pseudo-Eisenstein series, we would prefer a decomposition in terms of irreducibles. Following the GL_2 case, we will decompose the pseudo-Eisenstein series into genuine Eisenstein series. (Since Eisenstein series are images of principal series, they are eigenfunctions for Casimir and for the whole center of the universal enveloping algebra. Typically principal series are irreducible, so Eisenstein series typically generate irreducible representations.) Again, due to the plurality of parabolics in GL_3, we have several kinds of Eisenstein series in GL_3. The definition for GL_2 Eisenstein series given above scales nicely to include all of these: for a parabolic P, the P-Eisenstein series is

$$E_\chi = \sum_{\gamma \in P_k\backslash G_k} f_\chi(\gamma g)$$

where f_χ is a (spherical) vector in a representation χ of M^P, extended to a P-representation by left N-invariance, and induced up to G.

The key to obtaining the spectral decomposition for GL_2 pseudo-Eisenstein series is that the Levi component is a product of copies of GL_1, allowing us to reduce to the spectral theory for GL_1 (Mellin inversion). For GL_3 we are able to use a similar approach for *minimal* parabolic pseudo-Eisenstein series, again because the Levi component is a product of copies of GL_1. The same methods will certainly *not* work for decomposing $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series, because in these cases the Levi component contains a copy of GL_2.

So we turn our attention first to the decomposition of the minimal parabolic pseudo-Eisenstein series. We will need the functional equation of the Eisenstein series. Note that because of the increase in dimension, the symmetry of the Eisenstein series is more complex. The Eisenstein series can no longer be parameterized by one complex number s, since the data f_χ for the Eisenstein series is on a product of three copies of GL_1. The symmetries of the Eisenstein series can be described in terms of the action of the Weyl group W on the standard maximal torus A (which, in this case, is the same as the Levi component M) on its Lie algebra $\mathfrak{a}$, and the dual space $i\mathfrak{a}^*$. For now, we give a minimal explanation of this action, just enough to describe the constant term and the functional equations of the Eisenstein series and use them in the spectral decomposition. See the appendices for further details on the computation of constant terms and the derivation of the functional equations.

For GL_n the standard maximal torus A is the product of n copies of GL_1, and representations of A are products of representations of GL_1; in the unramified case, these representations are just $y \to y^{s_i}$, for complex s_i. The Weyl group W is the group of permutation matrices in GL_n. It acts on A by permuting the copies of GL_1, and it acts on the dual in the canonical way, permuting the s_i, in the unramified case.

We now describe the constant term and the functional equations of the Eisenstein series. The constant term of the Eisenstein series (along the minimal parabolic) has the form

$$c_P(E_\chi) = \sum_{w\in W} c_w(\chi)\cdot w\chi$$

where $w\chi$ is the image of χ under the action of w and $c_w(\chi)$ is a constant depending on w and χ with $c_1(\chi) = 1$. The Eisenstein series has functional equations

$$c_w(\chi)\cdot E_\chi = E_{w\chi} \quad \text{for all } w\in W$$

We start the decomposition of Ψ_φ by using the spectral expansion of its data φ. Recall that φ is left $N_{\mathbb{A}}$-invariant, so it is essentially a function on the Levi component, which is a product of copies of $k^\times\backslash\mathbb{J}$. (By Fujisaki's lemma, this is the product of a ray with a compact abelian group. To simplify the present discussion we will assume that the compact abelian group is trivial, as is the case for number fields with class number one, e.g. $k = \mathbb{Q}$.) So spectrally decomposing φ is a higher-dimensional version of Mellin inversion.

$$\varphi = \int \langle\varphi,\chi\rangle\cdot\chi\, d\chi$$

Winding up,

$$\Psi_\varphi(g) = \int_{i\mathfrak{a}^*}\langle\varphi,\chi\rangle\cdot E_\chi(g)\, d\chi$$

Note that in order for this to be valid, the parameters of χ must have $\mathrm{Re}(s_i)\gg 1$. However, in order to use the symmetries of the functional equations, we need the parameters to be on the critical line. In moving the contours, we pick up some residues, which fortunately are constants. Breaking up the dual space according to Weyl chambers and changing variables,

$$\Psi_\varphi(g)\ -\ (\text{residues}) = \sum_{w\in W}\int_{\text{1st Weyl chamber}}\langle\varphi, w\,\chi\rangle\cdot E_{w\,\chi}(g)\, d\chi$$

Now using the functional equations,

$$\begin{aligned}\Psi_\varphi(g)\ -\ (\text{residues}) \quad &= \quad \sum_{w\in W}\int_{(1)}\langle\varphi, w\,\chi\rangle\cdot c_w(\chi)\, E_\chi(g)\, d\chi \\ &= \quad \int_{(1)}\sum_{w\in W}\langle\varphi, c_w(\chi)\, w\,\chi\rangle\cdot E_\chi(g)\, d\chi\end{aligned}$$

We recognize the constant term of the Eisenstein series, and apply the adjointness relation

$$\sum_{w\in W}\langle\varphi, c_w(\chi)\, w\,\chi\rangle = \langle\varphi, c_P E_\chi\rangle = \langle\Psi_\varphi, E_\chi\rangle$$

So we have,

$$\Psi_\varphi(g) = \int_{(1)}\langle\Psi_\varphi, E_\chi\rangle\cdot E_\chi(g)\, d\chi\ +\ (\text{residues})$$

Our next goal is to show that the remaining automorphic forms, namely those with cuspidal support $P^{2,1}$ or $P^{1,2}$, can be written as superpositions of genuine $P^{2,1}$ Eisenstein series. To do this it suffices to decompose $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series with cuspidal support. For this discussion we let $P = P^{2,1}$ and $Q = P^{1,2}$.

We start by looking more carefully at pseudo-Eisenstein series with cuspidal data. The data for a P pseudo-Eisenstein series is smooth, compactly supported, and left $Z_{\mathbb{A}}M_k^P N_{\mathbb{A}}^P$-invariant. For now, we assume that the data is spherical, i.e. right K-invariant. This means that this function is determined by its behavior on $Z_{\mathbb{A}}M_k^P\backslash M_{\mathbb{A}}^P$. In contrast to the minimal parabolic case, this is *not* a product of copies of GL_1, so we cannot simply use the GL_1 spectral theory (Mellin inversion) to accomplish the decomposition. Instead, this quotient is isomorphic to $GL_2(k)\backslash GL_2(\mathbb{A})$, so we will use the spectral theory for GL_2. If η is the data for a $P^{2,1}$ pseudo-Eisenstein series Ψ_η, we can write η as a tensor product $f \otimes \nu$ on

$$Z_{GL_2(\mathbb{A})} GL_2(k)\backslash GL_2(\mathbb{A}) \,\cdot\, Z_{GL_2(k)}\backslash Z_{GL_2(\mathbb{A})}$$

Saying that the data is *cuspidal* means that f is a cusp form. Similarly the data $\varphi = \varphi_{F,s}$ for a $P^{2,1}$-Eisenstein series is the tensor product of a GL_2 cusp form F and a character $\chi_s = |\cdot|^s$ on GL_1. We show that $\Psi_{f,\nu}$ is the superposition of Eisenstein series $E_{F,s}$ where F ranges over an orthonormal basis of cusp forms and s is on a vertical line.

Using the spectral expansions of f and ν,

$$\eta = f \otimes \nu = \Big(\sum_{\text{cfms } F} \langle f, F\rangle \cdot F\Big) \cdot \Big(\int_s \langle \nu, \chi_s\rangle \cdot \chi_s \, ds\Big) = \sum_{\text{cfms } F} \int_s \langle \eta_{f,\nu}, \varphi_{F,s}\rangle \cdot \varphi_{F,s}\, ds$$

So the pseudo-Eisenstein series can be re-expressed as a superposition of Eisenstein series.

$$\begin{aligned}
\Psi_{f,\nu}(g) &= \sum_{\gamma \in P_k\backslash G_k} \eta_{f,\nu}(\gamma g) \\
&= \sum_{\gamma \in P_k\backslash G_k} \sum_{\text{cfms } F} \int_s \langle \eta_{f,\nu}, \varphi_{F,s}\rangle \cdot \varphi_{F,s}(\gamma g)\, ds \\
&= \sum_{\text{cfms } F} \int_s \langle \eta_{f,\nu}, \varphi_{F,s}\rangle \sum_{\gamma \in P_k\backslash G_k} \varphi_{F,s}(\gamma g)\, ds \\
&= \sum_{\text{cfms } F} \int_s \langle \eta_{f,\nu}, \varphi_{F,s}\rangle \cdot E_{F,s}(g)\, ds
\end{aligned}$$

In fact the coefficient $\langle \eta, \varphi\rangle_{GL_2}$ is the same as the pairing $\langle \Psi_\eta, E_\varphi\rangle_{GL_3}$, since

$$\langle \Psi_\eta, E_\varphi\rangle = \langle c_P(\Psi_\eta), \varphi\rangle = \langle \eta, \varphi\rangle$$

So the spectral expansion is

$$\Psi_{f,\nu} = \sum_{\text{cfms } F} \int_s \langle \Psi_{f,\nu}, E_{F,s}\rangle \cdot E_{F,s}(g)\, ds$$

Notice that, so far, we have not had to shift the line of integration to the critical line $\frac{1}{2} + i\mathbb{R}$.

It now remains to show that pseudo-Eisenstein series for the associate parabolic, $Q = P^{1,2}$, can also be decomposed into superpositions of P-Eisenstein series. Notice that in the discussion above, when we decomposed P-pseudo-Eisenstein series into genuine P-Eisenstein series, we did not use the functional equation to fold up the integral, as in the case of minimal parabolic pseudo-Eisenstein series. For maximal parabolic Eisenstein series, the functional equation does *not* relate the Eisenstein series to itself, but rather to the Eisenstein series of the associate parabolic. We will use this functional equation to obtain the decomposition of associate parabolic pseudo-Eisenstein series. For a derivation of the functional equation, see the appendix. For now, we state the functional equation without proof:

$$E^Q_{F,s} = b_{F,s} \cdot E^P_{F,1-s}$$

where $b_{f,s}$ is a meromorphic function that appears in the computation of the constant term along P of the Q-Eisenstein series.

We consider a Q-pseudo-Eisenstein series $\Psi^Q_{f,\nu}$ with cuspidal data. By the same arguments used above to obtain the decomposition of P-pseudo-Eisenstein series, we can decompose $\Psi^Q_{f,\nu}$ into a superposition of Q-Eisenstein series.

$$\Psi^Q_{f,\nu}(g) = \sum_{\text{cfms } F} \int_s \langle \eta_{f,\nu}, \varphi_{F,s}\rangle \cdot E^Q_{F,s}(g)$$

Now using the functional equation,

$$\Psi_{f,\nu}^{Q}(g) = \sum_{\text{cfms } F} \int_s \langle \Psi_{f,\nu}^{Q}, b_{F,s} \cdot E_{F,1-s}^{P} \rangle \cdot b_{F,s} \cdot E_{F,1-s}^{P} = \sum_{\text{cfms } F} \int_s \langle \Psi_{f,\nu}^{Q}, E_{F,1-s}^{P} \rangle \cdot |b_{F,s}|^2 \cdot E_{F,1-s}^{P}$$

So we have a decomposition of Q-pseudo-Eisenstein series (with cuspidal data) into P-Eisenstein series (with cuspidal data). In order to use the functional equation we did have to move some contours, but in this case there are no poles, so we do not pick up any residues.

We have described the spectral decomposition of $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})$ as the direct sum/integral of irreducibles. Any automorphic form ξ can be written as

$$\xi = \sum_{GL_3 \text{ cfms } f} \langle \xi, f \rangle \cdot f \;\; + \sum_{GL_2 \text{ cfms } F} \int_s \langle \xi, E_{F,s}^{2,1} \rangle \cdot E_{F,s}^{2,1} \;\; + \int_{(1)} \langle \xi, E_{\chi}^{1,1,1} \rangle \cdot E_{\chi}^{1,1,1} \, d\chi \;\; + \;\; \frac{\langle \xi, 1 \rangle}{\langle 1, 1 \rangle}$$

Certainly this expansion converges in L^2. To ensure more convergence, for example uniform convergence on compact sets, additional conditions need to be imposed. The arguments given above, proving the convergence of the SL_2 spectral expansion under sufficient differentiability conditions, generalize to GL_3.

A. Appendices

Here we include some supplemental material, which may serve as a useful addendum to the discussions above. In the first appendix, we provide the computations of some constant terms for GL_3 Eisenstein series using the Bruhat decomposition. The second appendix includes the derivation of the functional equations for GL_3 Eisenstein series from their constant terms.

A.1 Constant Terms of GL_3 Eisenstein Series

Since the constant terms of GL_3 Eisenstein series were used repeatedly in the discussion of the spectral decomposition of GL_3, we briefly discuss the way to obtain constant terms using the Bruhat decomposition. Recall the Bruhat decomposition of GL_n

$$G = \bigcup_{w \in W} PwQ = \bigsqcup_{w \in (W \cap P) \backslash W / (W \cap Q)} PwQ$$

where W is the Weyl group and P and Q are parabolics.

To compute the constant term along P of a Q-Eisenstein series,

$$\begin{aligned}
c_P(E_{\varphi}^{Q})(g) &= \int_{N_k^P \backslash N_{\mathbb{A}}^P} \sum_{\gamma \in Q_k \backslash G_k / P_k} \sum_{\beta \in Q_k \backslash Q_k \gamma P_k} \varphi(\gamma \beta n g) \, dn \\
&= \sum_{\gamma \in Q_k \backslash G_k / P_k} \int_{N_k^P \backslash N_{\mathbb{A}}^P} \sum_{\beta \in Q_k \backslash Q_k \gamma P_k} \varphi(\gamma \beta n g) \, dn \\
&= \sum_{w \in (W \cap P) \backslash W / (W \cap Q)} \int_{N_k^P \backslash N_{\mathbb{A}}^P} \sum_{\beta \in Q_k \backslash Q_k w P_k} \varphi(w \beta n g) \, dn \\
&= \sum_{w \in (W \cap P) \backslash W / (W \cap Q)} \int_{N_k^P \backslash N_{\mathbb{A}}^P} \sum_{\beta \in (w^{-1} Q_k w \cap P_k) \backslash P_k} \varphi(w \beta n g) \, dn
\end{aligned}$$

Further computation is dependent on the choice of P and Q. We show the computations for several of the constant terms for GL_3 Eisenstein series.

First consider $P = Q = P^{1,1,1}$ the minimal parabolic. Then the constant term is of the form:

$$c_{1,1,1}(E_{\varphi}^{1,1,1}) = \sum_{w \in W} c_w(\chi) \, w\chi \qquad \text{where } c_1(\chi) = 1$$

when φ is in the principal series I_χ. We recall the computations that yield this conclusion.

The double coset space $(W \cap P)\backslash W/(W \cap P)$ is the whole Weyl group W, and since the Levi component is invariant under conjugation by elements of W, $PwP = PwN$ for all w. So the constant term is

$$c_{1,1,1}(E_\varphi^{1,1,1})(g) = \sum_{w \in W} \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\beta \in (w^{-1}P_k w \cap N_k)\backslash N_k} \varphi(w\beta ng)\, dn$$

For $w = 1$,

$$\int_{N_k\backslash N_{\mathbb{A}}} \varphi(ng)\, dn = \mathrm{vol}(N_k\backslash N_{\mathbb{A}}) \cdot \varphi(g) = \varphi(g)$$

and for $w = w_o$, the long Weyl element, the intersection $w_o^{-1}P_k w_o \cap N_k$ is trivial, so there is unwinding

$$\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma \in N_k} \varphi(w_o\gamma ng)\, dn = \int_{N_{\mathbb{A}}} \varphi(w_o ng)\, dn$$

and this integral factors over primes because φ does.

The integrals corresponding to the four other elements of the Weyl group have partial unwinding. First consider $w = \sigma$, the element corresponding to the reflection of the first positive simple root. Then the quotient $(\sigma^{-1}N_k\sigma \cap N_k)\backslash N_k$ is isomorphic to the GL_2 unipotent radical, here denoted $N^{1,1}$. So the integral is

$$\begin{aligned}
\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma \in (\sigma^{-1}N_k\sigma \cap N_k)\backslash N_k} \varphi(\sigma\gamma ng)\, dn &= \int_{N_k^{1,1}\backslash N_{\mathbb{A}}^{1,1}} \int_{N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}} \sum_{\gamma \in N_k^{1,1}} \varphi(\sigma\gamma nug)\, du\, dn \\
&= \int_{N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}} \int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma nug)\, dn\, du \\
&= \int_{N_{\mathbb{A}}^{1,1}} \int_{N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}} \varphi(u\sigma ng)\, du\, dn \\
&= \mathrm{vol}(N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}) \times \int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma ng)\, dn \\
&= \int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma ng)\, dn
\end{aligned}$$

This computation relies on the facts that $\sigma N^{2,1}\sigma^{-1} = N^{2,1}$ and φ is $N^{2,1}$-invariant. This last integral factors over primes.

We can compute the terms corresponding to the other Weyl elements similarly. For $w = \tau$, the element corresponding to the reflection of the second positive simple root,

$$\int_{N_{\mathbb{A}}^{1,1}} \varphi(\tau ng)\, dn$$

For $w = \tau\sigma$,

$$\int_{N_{\mathbb{A}}^{2,1}} \varphi(\tau\sigma ng)\, dn$$

Finally, for $w = \sigma\tau$,

$$\int_{N_{\mathbb{A}}^{1,2}} \varphi(\sigma\tau ng)\, dn$$

These integrals factor over primes, and the local integrals are intertwining operators among principal series: $T_{w,\chi_v} : I_{\chi_v} \to I_{w\chi_v}$. For example, consider the local integral for $w = \sigma$. Using right K_v-invariance,

$$T_{w,\chi_v}\varphi_v(g) = \int_{N_v} \varphi_v(\sigma ng)\, dn = \int_{N_v} \varphi_v(\sigma nn_g m_g)\, dn = \int_{N_v} \varphi_v(\sigma nm_g)\, dn$$

Changing variables $n \to m_g n m_g^{-1}$ and using the P-equivariance of φ_v by χ_v,

$$T_{w,\chi_v}\varphi_v(g) = \delta(m_g)\int_{N_v}\varphi_v(\sigma m_g n)\,dn = \delta(m_g)\int_{N_v}\chi_v(\sigma m_g \sigma^{-1})\varphi_v(\sigma n)\,dn$$

Notice that this is the action of W on χ_v, so

$$T_{w,\chi_v}\varphi_v(g) \;=\; \delta(m_g)\cdot\sigma\chi_v(m_g)\int_{N_v}\varphi_v(\sigma n)\,dn \;=\; \delta(m_g)\cdot\sigma\chi_v(m_g)\cdot T_{\sigma,\chi_v}\varphi_v(1)$$

So the constant term is

$$c_{1,1,1}(E_\varphi^{1,1,1})(g) = \sum_{w\in W}\left(\prod_v T_{w,\chi_v}\varphi_v(1)\right)\cdot\delta(m_g)\cdot w\chi(m_g)$$

Defining $c_w(\chi)$ to be the constant in front and renormalizing to eliminate the modular function, we obtain the desired expression for the constant term:

$$c_{1,1,1}(E_\varphi^{1,1,1})(g) = \sum_{w\in W} c_w(\chi)\cdot w\chi(g)$$

Now we consider the case where P is the minimal parabolic and Q is one of the maximal parabolics, say $P^{2,1}$. We consider Q-Eisenstein series with cuspidal data. The constant term $c_{1,1,1}(E_\varphi^{2,1})$ is identically zero. To see how this can be computed, recall from above,

$$c_{1,1,1}(E_\varphi^{2,1})(g) = \sum_{w\in(W\cap P)\backslash W/(W\cap Q)}\int_{N_k^P\backslash N_{\mathbb{A}}^P}\sum_{\beta\in(w^{-1}Q_k w\cap P_k)\backslash P_k}\varphi(w\beta n g)\,dn$$

As in the previous case, the Levi component of P is invariant under conjugation by W so $QwP = QwN$, where N denotes the unipotent radical of P. The quotient $(W\cap P)\backslash W/(W\cap Q)$ has three distinct cosets, with representatives $w = 1, \tau, \tau\sigma$. So,

$$c_{1,1,1}(E_\varphi^{2,1}) = \sum_{w=1,\tau,\tau\sigma}\int_{N_k\backslash N_{\mathbb{A}}}\sum_{\beta\in(w^{-1}Q_k w\cap N_k)\backslash N_k}\varphi(w\beta n g)\,dn$$

For $w = 1$, the integral is

$$\begin{aligned}\int_{N_k\backslash N_{\mathbb{A}}}\varphi(ng)\,dn &= \int_{N_k^{1,1}\backslash N_{\mathbb{A}}^{1,1}}\int_{N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}}\varphi(nug)\,du\,dn\\ &= \mathrm{vol}(N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1})\times\int_{N_k^{1,1}\backslash N_{\mathbb{A}}^{1,1}}\varphi(ng)\,dn\end{aligned}$$

which is zero because φ is cuspidal. Similar computations show that the other two terms are zero as well.

Next we discuss the case where $P = Q$ is a maximal parabolic, say $P^{2,1}$. If the data φ for the Eisenstein series is cuspidal, the constant term is just φ. From the initial computations,

$$c_{2,1}(E_\varphi^{2,1})(g) = \sum_{w\in(W\cap P)\backslash W/(W\cap P)}\int_{N_k\backslash N_{\mathbb{A}}}\sum_{\beta\in(w^{-1}P_k w\cap P_k)\backslash P_k}\varphi(w\beta n g)\,dn$$

In this case, there are two double cosets, with representatives 1 and τ so,

$$c_{2,1}(E_\varphi^{2,1})(m) = \int_{N_k\backslash N_{\mathbb{A}}}\varphi(nm)\,dn \;+\; \int_{N_k\backslash N_{\mathbb{A}}}\sum_{\beta\in(\tau^{-1}P_k\tau\cap P_k)\backslash P_k}\varphi(\tau\beta n m)\,dn$$

Since φ is left N-invariant, the first term is just $\mathrm{vol}(N_k\backslash N_{\mathbb{A}})\cdot\varphi(m)$. Showing that the second term is zero takes a little more work. The quotient $(\tau^{-1}P_k\tau\cap P_k)\backslash P_k$ is the semidirect product of quotients of M_k and N_k, so the sum over β can be written as a double sum over the M_k part, a quotient isomorphic to $P_{GL_2(k)}\backslash GL_2(k)$, and the N_k part, a quotient by the unipotent radical U_k^1 which is zero in the $(2,3)$ entry.

$$\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\beta\in(\tau^{-1}P_k\tau\cap P_k)\backslash P_k} \varphi(\tau\beta nm)\,dn = \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\mu\in P_{GL_2(k)}\backslash GL_2(k)} \sum_{\nu\in U_k^1\backslash N_k} \varphi(\tau\mu\nu ng)\,dn$$

The sum over μ comes out of the integral, and the sum over ν unwinds yielding

$$\sum_{\mu\in P_{GL_2(k)}\backslash GL_2(k)} \int_{U_k^1\backslash N_{\mathbb{A}}} \varphi(\tau\mu nm)\,dn$$

Since M normalizes N, a change of variables eliminates the μ, while introducing the modular function. Letting U^2 be the unipotent radical such that $N^{2,1} = U^1 \times U^2$, the integral becomes

$$\sum_{\mu\in P_{GL_2(k)}\backslash GL_2(k)} \delta(\mu)\cdot \int_{U_{\mathbb{A}}^2} \int_{U_k^1\backslash U_{\mathbb{A}}^1} \varphi(\tau u_1 u_2 g)\,du_1\,du_2$$

Since τU^1 is a GL_2 unipotent radical, the inner integral is a GL_2 constant term, and this is zero, because of the cuspidality of φ. So the $P^{2,1}$ constant term for a $P^{2,1}$-Eisenstein series with cuspidal data is just equal to the term coming from the identity coset.

$$c_{2,1}(\Psi_\varphi^{2,1}) = \mathrm{vol}(N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1})\cdot\varphi$$

Finally consider the case where P and Q are the associate (maximal) parabolics, say $P = P^{2,1}$ and $Q = P^{1,2}$. We describe the constant terms, but omit the computations. Let E_φ^P be a P-Eisenstein series with cuspidal data φ^P on M^P,

$$\varphi^P(m) = \varphi_{f,s}^P\begin{pmatrix} A & * \\ & 1\end{pmatrix} = f(A)\cdot|\det A|^s$$

where f is a GL_2 cusp form and s is a complex number. Associated to this P-Eisenstein series is a Q-Eisenstein series E_φ^Q with data φ^Q on M^Q,

$$\varphi^Q(m) = \varphi_{f,s}^Q\begin{pmatrix} 1 & * \\ & A\end{pmatrix} = f(A)\cdot|\det A|^{-s}$$

Then the constant term along Q of the P-Eisenstein series is of the form

$$c_P(E_\varphi^Q) = c_{2,1}(E_\varphi^{1,2}) = a_{f,s}\cdot\varphi_{f,1-s}^Q$$

and similarly,

$$c_Q(E_\varphi^P) = c_{1,2}(E_\varphi^{2,1}) = b_{f,s}\cdot\varphi_{f,1-s}^P$$

The coefficients $a_{f,s}$ and $b_{f,s}$ are meromorphic functions of s.

A.2 Functional Equations of GL_3 Eisenstein Series

Here we recall the derivation of the functional equations for GL_3 Eisenstein series from their constant terms. We set the discussion in GL_n, since the same arguments work and are in fact clearer.

The functional equations for minimal parabolic spherical Eisenstein series are of the form

$$E_\chi = A(\chi,w)\cdot E_{w\chi} \quad \text{for all } w\in W$$

The existence of such equations follows from the functional equation of the GL_2 Eisenstein series. The key is that a GL_n minimal parabolic Eisenstein series is the composition of an Eisenstein series for a next-to-minimal parabolic Q with something isomorphic to a GL_2 Eisenstein series.

Since the Weyl group is generated by reflections, it suffices to consider intertwining operators given by a simple reflection: let σ be the Weyl element that flips the ith positive root, and fixes all other positive roots. The corresponding next-to-minimal parabolic, Q is strictly upper triangular, except for the $(i, i+1)^{\text{th}}$ entry. The quotient $P_k\backslash G_k$ is the direct sum of $Q_k\backslash G_k$ and $P_k\backslash Q_k$, which is a copy of the GL_2 quotient $P_{GL_2}(k)\backslash GL_2(k)$. So the Eisenstein series is an iterated sum:

$$E_\chi(g) = \sum_{\gamma \in Q_k\backslash G_k} \sum_{\delta \in P_k\backslash Q_k} f(\delta\gamma g)$$

Consider the subgroup H of GL_n isomorphic to GL_2 that has entries in the two-by-two block starting at the $(i, i)^{\text{th}}$ entry. For fixed g in $G = GL_n$, the map f_g on H given by $h \longrightarrow f(hg)$ is in the GL_2 principal series I_χ (where χ is restricted to $P \cap H$). So, for fixed $g \in G$ the series

$$\tilde{E}_{\chi,g}(h) = \sum_{\delta \in P_k\backslash Q_k} f_g(\delta h)$$

is a GL_2 Eisenstein series. Parameterize χ by $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$. Then the action of σ on χ interchanges s_i and s_{i+1}. In the GL_2 case, we usually take a quotient by the center, which enables us to parameterize χ by one $s \in \mathbb{C}$, and the action of σ is $s \to 1 - s$. The familiar functional equation of GL_2 Eisenstein series can be restated as

$$E_\chi(h) = A(\chi, \sigma) \cdot E_{\sigma\chi}(h)$$

Applying this to the iterated Eisenstein series, we obtain the functional equations for GL_n minimal-parabolic Eisenstein series.

$$E_\chi(g) = \sum_{\gamma \in Q_k\backslash G_k} \tilde{E}_{\chi,\gamma g}(h) \quad = \quad A(\chi, \sigma) \sum_{\gamma \in Q_k\backslash G_k} \tilde{E}_{\sigma\chi,\gamma g}(h) \quad = \quad A(\chi, \sigma) \cdot E_{\sigma\chi}(g)$$

Now we recall the way to obtain the constants $A(\chi, w)$ from the constant term of E_χ along P,

$$c_P(E_\chi) = \sum_{\sigma \in W} c_\sigma(\chi) \cdot \sigma\chi$$

Taking the constant term of both sides of the functional equation yields:

$$\begin{aligned} c_P(E_\chi) &= A(\chi, w) \cdot c_P(E_{w\chi}) \\ \sum_{\sigma \in W} c_\sigma(\chi) \cdot \sigma\chi &= A(\chi, w) \sum_{\sigma \in W} c_\sigma(w\chi) \cdot \sigma w\chi \\ \sum_{\sigma \in W} c_{\sigma w}(\chi) \cdot \sigma w\chi &= \sum_{\sigma \in W} A(\chi, w) \cdot c_\sigma(w\chi) \cdot \sigma w\chi \end{aligned}$$

Since the $w\chi$ are linearly independent,

$$c_{\sigma w}(\chi) = A(\chi, w) \cdot c_\sigma(w\chi)$$

So, for all $\sigma, w \in W$,

$$A(\chi, w) = \frac{c_{\sigma w}(\chi)}{c_\sigma(w\chi)}$$

and, in particular, setting $\sigma = 1$ gives

$$A(\chi, w) = c_w(\chi)$$

So the functional equation becomes

$$E_\chi = c_w(\chi) \cdot E_{w\chi}$$

Next we discuss the derivation of the functional equations for *maximal* parabolic Eisenstein series from their constant terms. This argument parallels the argument for GL_2, hinging on the facts that: (1) apart from their constant terms, automorphic forms are of rapid decay on Siegel sets and (2) the maximal parabolic Eisenstein series have the same Casimir eigenvalues. Using the constant terms described above, in Siegel sets,

$$\begin{aligned} E^P_{f,s} &= \varphi^P_{f,s} + a_{f,s} \cdot \varphi^Q_{f,1-s} + \text{(rapid decay)} \\ E^Q_{f,s} &= \varphi^Q_{f,s} + b_{f,s} \cdot \varphi^P_{f,1-s} + \text{(rapid decay)} \end{aligned}$$

Manipulate the second equation to obtain cancellation: send $s \to 1-s$ and divide by $b_{f,1-s}$.

$$\frac{1}{b_{f,1-s}} \cdot E^Q_{f,1-s} = \varphi^P_{f,s} + \frac{1}{b_{f,1-s}} \cdot \varphi^Q_{f,1-s} + \text{(rapid decay)}$$

Now subtracting from $E^P_{f,s}$,

$$E^P_{f,s} - \frac{1}{b_{f,1-s}} \cdot E^Q_{f,1-s} = \big(a_{f,s} - \frac{1}{b_{f,1-s}}\big) \cdot \varphi^Q_{f,1-s} + \text{(rapid decay)}$$

For $\mathrm{Re}(1-s) \gg 0$, this difference is in L^2, and it is an eigenfunction for Casimir. By the self-adjointness of Casimir, the eigenvalue must be negative real. However, there is a continuum of eigenvalues for which this is not true, and so the identity principle implies that the difference is identically zero, proving the functional equation.

$$E^P_{f,s} = \frac{1}{b_{f,1-s}} \cdot E^Q_{f,1-s}$$