Integral Representations of L-functions

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Here we briefly recall the integral representations of $GL_n \times GL_m$ L-functions, situating them in the context of spectral decompositions. We start with the familiar $GL_2 \times GL_1$ case. For a GL_2 automorphic form f, consider the Iwasawa-Tate zeta integral

$$\int_{k^\times \backslash \mathbb{J}} f\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \chi(y)\, |y|^{s-1/2}\, dy$$

(This is the modernized, Jacquet-Langlands version of Hecke's Mellin transform.) Observe that this can be viewed as the $(\chi, s)^{\text{th}}$ decomposition coefficient for the L^2 spectral expansion of f restricted to GL_1. By Fujisaki's lemma, $k^\times \backslash \mathbb{J}$ is isomorphic to the product of a compact abelian group with the ray $(0, \infty)$. So the characters on $k^\times \backslash \mathbb{J}$ are tensor products $\chi \otimes |\cdot|^s$, where χ is (a grossencharacter) on the compact abelian part, and $|\cdot|^s$, for complex s, is a character on the ray.

The fact that this integral represents an L-function is made clear using Fourier expansion and the uniqueness of Whittaker models.

Recall that we obtain the Fourier-Whittaker expansion of f from the expansion of $R_g f$ along $N_k \backslash N_{\mathbb{A}} \approx k \backslash \mathbb{A}$,

$$\begin{aligned} R_g f(n) &= \sum_{\psi} \langle R_g f, \psi \rangle \cdot \psi(n) \\ &= \sum_{\xi \in k^\times} \langle R_g f, \psi_\xi \rangle \cdot \psi_\xi(n) \\ &= \sum_{\xi \in k^\times} W_f(m_\xi g) \cdot \psi_1(m_\xi^{-1} n m_\xi) \end{aligned}$$

So, letting $n = 1$, we get:

$$f(g) = R_g(1) = \sum_{\xi \in k^\times} W_f(m_\xi g)$$

Inserting the Fourier-Whittaker expansion to the integral,

$$\begin{aligned} \int_{k^\times \backslash \mathbb{J}} f\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \cdot \chi(y) \cdot |y|^{s-1/2}\, dy &= \int_{k^\times \backslash \mathbb{J}} \sum_{\xi \in k^\times} W_f\left(\left(\begin{smallmatrix} \xi & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)\right) \cdot \chi(y) \cdot |y|^{s-1/2}\, dy \\ &= \int_{\mathbb{J}} W_f\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \cdot \chi(y) \cdot |y|^{s-1/2}\, dy \end{aligned}$$

When f generates an irreducible admissible representation, the Whittaker function factors over primes, so we obtain the factorization into local integrals.

$$\int_{\mathbb{J}} W_f\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \cdot \chi(y) \cdot |y|^{s-1/2}\, dy = \prod_v \int_{k_v} W_{f,v}\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \cdot \chi_v(y) \cdot |y|_v^{s-1/2}\, dy$$

This argument generalizes to $GL_n \times GL_{n-1}$. Let f be a GL_n automorphic form f, generating an irreducible admissible representation. Restricting f to $H = GL_{n-1}$ and taking the F^{th} spectral component (where F is a GL_{n-1} automorphic form) yields the integral

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} f(h)\, \bar{F}(h)\, dh$$

Using the Fourier expansion of f, the uniqueness of Whittaker models ensures that this integral factors over primes.

Next we review integral representations of $GL_n \times GL_n$ L-functions. We will discuss the $GL_2 \times GL_2$ case, which is the familiar Rankin-Selberg convolution. The same arguments that work for GL_2 work for GL_n.

$$L(s, f_1 \times f_2) = \int_{Z_{\mathbb{A}} GL_2(k) \backslash GL_2(\mathbb{A})} f_1(g)\, f_2(g)\, E_s(g)\, dg$$

This factors into local integrals. We start by unwinding the Eisenstein series.

$$\begin{aligned}
&\int_{Z_{\mathbb{A}} GL_2(k) \backslash GL_2(\mathbb{A})} f_1(g) \cdot f_2(g) \cdot E_s(g)\, dg \\
&\quad = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} f_1(g) \cdot f_2(g) \cdot \sum_{\gamma \in P_k \backslash G_k} |\det(\gamma \cdot g)|^s\, dg \\
&\quad = \int_{Z_{\mathbb{A}} P_k \backslash G_{\mathbb{A}}} f_1(g) \cdot f_2(g) \cdot |\det(g)|^s\, dg \\
&\quad = \int_{Z_{\mathbb{A}} M_k \backslash M_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} f_1(mn) \cdot f_2(mn) \cdot |\det(m)|^s\, dn\, dm
\end{aligned}$$

Use the Fourier-Whittaker expansion for f_1,

$$\begin{aligned}
&\int_{Z_{\mathbb{A}} M_k \backslash M_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} f_1(mn) \cdot f_2(mn) \cdot |\det(m)|^s\, dn\, dm \\
&\quad = \int_{Z_{\mathbb{A}} M_k \backslash M_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} \sum_{\xi \in k^\times} W_{f_1}(m_\xi mn) \cdot f_2(mn) \cdot |\det(m)|^s\, dn\, dm \\
&\quad = \int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} W_{f_1}(mn) \cdot f_2(mn) \cdot |\det(m)|^s\, dn\, dm
\end{aligned}$$

Change variables: $n \to m^{-1} n m$

$$\begin{aligned}
&\int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} W_{f_1}(mn) \cdot f_2(mn) \cdot |\det(m)|^s\, dn\, dm \\
&\quad = \int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} W_{f_1}(nm) \cdot f_2(nm) \cdot |\det(m)|^s \cdot \delta^{-1}(m)\, dn\, dm \\
&\quad = \int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} W_{f_1}(m) \left(\int_{N_k \backslash N_{\mathbb{A}}} \psi(n) \cdot f_2(nm)\, dn \right) |\det(m)|^s \cdot \delta^{-1}(m)\, dm \\
&\quad = \int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} W_{f_1}(m) \cdot W_{f_2}(m) \cdot |\det(m)|^s \cdot \delta^{-1}(m)\, dm
\end{aligned}$$

This integral factors into local integrals,

$$\prod_v \int_{k_v} W_{f_1,v}\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot W_{f_2,v}\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot |y|_v^s \, \frac{dy_v}{y}$$

The L-function has a pole at $s = 1$, due to the Eisenstein series, and the residue is

$$\int_{Z_{\mathbb{A}} GL_2(k) \backslash GL_2(\mathbb{A})} f_1(g)\, f_2(g)\, |\det(g)|^s\, dg$$