# Spectral identities and exact formulas for counting lattice points in symmetric spaces

Amy DeCelles

University of Minnesota

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# Outline

# Motivation: Lattice-point counting in Euclidean spaces

In $\mathbb{R}^3$

$$N(T) = \#\{\xi \in \mathbb{Z}^3 : |\xi| \leq T\} = c \cdot T^3 + \mathrm{O}(T^2)$$

- dominant term: *volume* of sphere, error term: *surface area*
- failure in hyperbolic spaces: volume proportionate to surface area

# In $G/K$: spectral methods, an exact formula

- Spectral methods
  - instead of packing arguments
- Exact formula
  - instead of asymptotic formula
  - e.g. Riemann-von-Mangoldt
  - here: sum over lattice points and sum over automorphic spectrum
- Obtain exact formula from spectral identity using residue calculus (Perron-like identity).

# Simple case first: hyperbolic 3-space

- Hyperbolic 3-space: $G/K = SL_2(\mathbb{C})/SU(2)$
  - $SL_2(\mathbb{C})$ is rank one
  - $SL_2(\mathbb{C})$ is complex
- Lattice: $\Gamma \subset SL_2(\mathbb{C})$, such that $\Gamma \backslash SL_2(\mathbb{C})$ is *compact*.

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- Lattice: $\Gamma \subset SL_2(\mathbb{C})$, such that $\Gamma \backslash SL_2(\mathbb{C})$ is *compact*.
- Basepoint: $z_o = 1 \cdot K$
- Complex Parameter: $s \in \mathbb{C}$

# Outline

# Spectral identity

$$\sum_{\gamma \in \Gamma} \frac{r_\gamma e^{-(2s-1)r_\gamma}}{(2s-1)\sinh r_\gamma} = \sum_F \frac{F(z_o)^2}{(s_F(s_F-1) - s(s-1))^2}$$

where

- $r_\gamma = d(\gamma z_o, z_o)$
- $F$ ranges over an orthonormal basis of automorphic forms
- $s_F(s_F-1) \in \mathbb{C}$ is the eigenvalue of Casimir on $F$

# Apply integral transform

$$f \longrightarrow \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} f(s) \frac{(2s-1)e^{2sX}}{s(s+1)(s+2)}\, ds$$

$$\text{Left side} \longrightarrow \sum_{\gamma : r_\gamma < X} r_\gamma (1 + e^{-2(X - r_\gamma)})^2$$

$$\text{Right side} \longrightarrow \sum_F F(z_o)^2 \cdot e^{2X s_F} \cdot P_X(s_F)$$

where $P_X$ is an explicit rational function in $s_F$.

# The exact formula

Putting them together

$$\sum_{\gamma : r_\gamma < X} r_\gamma (1 + e^{-2(X - r_\gamma)})^2 = \sum_F F(z_o)^2 \cdot e^{2X s_F} \cdot P_X(s_F)$$

smoothed counting

automorphic spectrum

# Outline

# Obtaining the spectral identity

- automorphic PDE's (instead of e.g. RTF)

$$(\Delta - \lambda)^2 \, u_\lambda^{\mathsf{afc}} \; = \delta_{z_o}^{\mathsf{afc}}$$

  - $\lambda = s(s-1) \in \mathbb{C}$
  - Laplacian $\Delta$: Casimir operator descended to $G/K$.
  - $\Delta$ left $G$-invariant $\Rightarrow$ descends to $\Gamma\backslash G/K$
  - $\delta_{z_o}^{\mathsf{afc}}$: automorphic delta function at $z_o$

# Strategy

Find two different expressions for solution $u_\lambda^{\mathsf{afc}}$.

- Directly: harmonic analysis on $\Gamma\backslash G$
- "Automorphizing" a free-space solution:
  - $\delta_{z_o}$ bi-$K$-invariant, so harmonic analysis on $K\backslash G/K$ gives free-space solution
  - average over left $\Gamma$-translates to get automorphic solution

# Harmonic analysis of automorphic forms

- $\Gamma$ compact $\Rightarrow$ no continuous spectrum

$$f = \sum_F \langle f, F \rangle \cdot F$$

- For automorphic delta,

$$\langle \delta_{z_o}^{\mathsf{afc}}, F \rangle = F(z_o)$$

# First expression for automorphic fundamental solution

- $F$ an eigenfunction for $\Delta \Rightarrow$

$$u_\lambda^{\mathsf{afc}} = \sum_F \frac{F(z_o)}{(\lambda_F - \lambda)^2} \cdot F \;, \quad \lambda_F = \text{eigenvalue of } \Delta \text{ on } F$$

- Apply at $z_o$:

$$u_\lambda^{\mathsf{afc}}(z_o) = \sum_F \frac{F(z_o)^2}{(s_F(s_F - 1) - s(s-1))^2}$$

# Harmonic analysis of bi-$K$-invariant functions

- $K\backslash G/K \approx A^+ \approx (0, +\infty)$
- Berezin-HC transform

$$\tilde{f}(\tfrac{1}{2} + i\xi) = \int_0^\infty f(a_r)\, \varphi_{\frac{1}{2}-i\xi}(a_r)\, \sinh^2 r\, dr$$

- Inversion

$$f = \int_{-\infty}^\infty \tilde{f}(\tfrac{1}{2} + i\xi)\, \varphi_{\frac{1}{2}+i\xi}\, |\xi|^2\, d\xi$$

- Spherical functions

$$\varphi_s(a_r) = \frac{\sinh((2s-1)r)}{(2s-1)\sinh r}$$

# Free-space solution

$$(\Delta - \lambda)^2 \, v_\lambda = \delta_{z_o}$$

- $\delta_{z_o}$ bi-$K$-invariant

$$\tilde{\delta}_{z_o} = \varphi_{\frac{1}{2}-i\xi}(1) = 1$$

- $\varphi_s$ an eigenfunction for $\Delta$ with eigenvalue $\lambda_s = s(s-1)$

$$\Rightarrow v_\lambda = \int_{-\infty}^{\infty} \frac{1}{(\lambda_{\frac{1}{2}+i\xi} - \lambda)^2} \cdot \varphi_{\frac{1}{2}+i\xi} \, |\xi|^{-2} \, d\xi$$

- Residue calculus $\Rightarrow$

$$v_\lambda(a_r) = \frac{r e^{-(2s-1)r}}{(2s-1)\sinh r}, \quad \lambda = s(s-1)$$

# Second expression for automorphic fundamental solution

Automorphize:

$$v_\lambda^{\mathsf{afc}} = \sum_{\gamma \in \Gamma} \gamma \circ v_\lambda$$

Apply at $z_o$ $(a_r = a_o = 1)$:

$$v_\lambda^{\mathsf{afc}}(z_o) = \sum_{\gamma \in \Gamma} \frac{r_\gamma e^{-(2s-1) r_\gamma}}{(2s-1)\sinh r_\gamma}\,, \quad r_\gamma = d(\gamma z_o, z_o)$$

# Spectral identity

Equate two expressions for automorphic fundamental solution:

$$v_\lambda^{\mathsf{afc}} = u_\lambda^{\mathsf{afc}}$$

$$\sum_{\gamma\in\Gamma} \frac{r_\gamma e^{-(2s-1)r_\gamma}}{(2s-1)\sinh r_\gamma} = \sum_F \frac{F(z_o)^2}{(s_F(s_F-1)-s(s-1))^2}$$

# Outline

# Lattice points in symmetric spaces

What is needed to do this for more general $G/K$? e.g. $G = GL_3$?

- spectral identity
- integral transform

# Spectral identity

To obtain the spectral identity we need

- harmonic analysis of automorphic forms
- harmonic analysis of bi-$K$-invariant functions

# Harmonic analysis of automorphic forms

For $GL_3$,

$$\begin{aligned} f \;=\; & \sum_{GL_3 \text{ cfms } F} \langle f, F\rangle \cdot F \;+\; \langle f, 1\rangle \\ & + \int_{\mathfrak{a}_+^*} \langle f, E_\chi^{\mathrm{min}}\rangle \cdot E_\chi^{\mathrm{min}} \, d\chi \\ & + \sum_{GL_2 \text{ cfms } \phi} \int_{\frac{1}{2}+i\mathbb{R}} \langle f, E_{s,\phi}^{2,1}\rangle \cdot E_{s,\phi}^{2,1} \, ds \end{aligned}$$

## Spherical transform, inversion

- $K\backslash G/K \approx A^+ \approx$ (product of rays)
- Spherical transform

$$\tilde{f} = \int_G f \cdot \overline{\varphi}_{\rho+i\xi} \, dg$$

where $\rho$ is the half sum of positive roots, $\xi \in \mathfrak{a}^*$

- Spherical inversion

$$f = \int_{\rho+i\mathfrak{a}^*} \tilde{f} \cdot \varphi_{\rho+i\xi} \cdot |\mathbf{c}(\rho+i\xi)|^{-2} \, d\xi$$

# Zonal spherical functions

- Complex groups: spherical functions are elementary.
- $\lambda^{\text{th}}$ spherical function ($\lambda \in \mathfrak{a}^*$)

$$\varphi_\lambda = \frac{\pi(\rho)}{\pi(i\lambda)} \cdot \frac{\sum \det(w) \cdot e^{wi\lambda}}{\sum \det(w) \cdot e^{w\rho}}$$

  - summing over $w \in$ Weyl group (permutations), so $\det(w)$ is just $\pm 1$
  - $\pi(\lambda)$ is the product of all $\langle \alpha, \lambda \rangle$'s, for positive roots $\alpha$
- The factor out front is the $\mathbf{c}$-function.

# Spectral identity for $G/K$

To get spectral identity:

- construct automorphic fundamental solution $u_\lambda^{\mathsf{afc}}$
- construct free-space fundamental solution $v_\lambda$ and automorphize: $v_\lambda^{\mathsf{afc}}$

But: explicit expression for free-space fundamental solution?

- spherical inversion is an integral over several complex parameters

Also: choices for differential operator

- Casimir does not generate the center of the universal enveloping algebra.

# Integral transform?

In simple case,

- integral transform: counting info from spectral identity
  - FT of $\frac{1}{x}$ is Heaviside
  - cut off terms outside certain radius

In higher rank,

- integral transform? ... FT of characteristic function of positive Weyl chamber?

# Further reading

Lattice-point counting (asymptotics)

- Patterson 1975, Levitan 1987, Lax-Phillips 1982, Bruggeman-Miatello-Wallach 1999, Gorodnik-Nevo 2010
- Iwaniec's book (*Spectral Methods* . . . , 2002)

Riemann-von-Mangoldt

- Freitag & Busam's book (*Complex Analysis*, 2005)

Spectral theory of automorphic forms

- books: Langlands (1976), Moeglin-Waldspurger (1995)

Zonal spherical functions

- books: Helgason (1984), Knapp (1986), Varadarajan (1989)

PDE's vs RTF

- Diaconu-Garrett 2009 (two papers)