Prime Numbers, Quantum Chaos, and Pseudo-Laplacians

Amy DeCelles

University of St. Thomas

St. Cloud State University
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Outline

See Jordan Ellenberg's book on the power of mathematical thinking for the classification of mathematics as either simple or complex and either shallow or profound, yielding four "quandrants" of mathematics: simple and shallow, simple and profound, complex and shallow, complex and profound. It is a fun read!

Primes: Simple and Profound

- Multiplicative building blocks
  - Every number can be (uniquely) written as the product of primes. (Unique prime factorization.)
  - Easier to multiply primes than to factor composite numbers. (Applications: crypto.)
- How many primes?
  - Infinitely many (Euclid).
  - How many have remainder 1 (or 3) after dividing by 4?
  - How many twin primes?
- How often do primes occur?

$$\pi(X) \;=\; \#\{p \text{ prime } \leqslant X\} \;\sim\; \frac{X}{\ln X} \qquad \text{(PNT)}$$

The Euler-Riemann Zeta Function

- Recall "p-series" from Calc II: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ but diverges for $p = 1$.
- Define zeta function: For a real number $s > 1$,

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1)$$

- Euler Product:

$$\zeta(s) \;=\; \prod_{p \text{ prime}} \frac{1}{1 - (1/p)^s}$$

$$\prod_{p \text{ prime}} \frac{1}{1-(1/p)^s} \qquad \text{(Euler Product)}$$

$$\overset{\text{Geom. Series}}{=} \prod_{p \text{ prime}} \left(\sum_{n=0}^{\infty} \left(\frac{1}{p^s}\right)^n \right)$$

$$= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \dots \right)$$

$$\cdot \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \dots \right)$$

$$\cdot \left(1 + \frac{1}{5^s} + \frac{1}{25^s} + \frac{1}{125^s} + \dots \right) \dots$$

$$\overset{\text{UPF!}}{=} 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \qquad \square$$

How many primes? (Analytic)

- Euler's (analytic!) proof that there are infintely many primes: If there are only finitely many primes, the Euler product for $\zeta(1)$ would be finite, but we know the harmonic series diverges!
- Modify to show that there are infinitely many primes of the form $a + bn$ for any fixed a and b that have no common factors. (Dirichlet: "infinitude of primes in arithmetic progressions.")
- Green-Tao 2004: Arbitrarily long arithmetic progressions in primes. (Largest found is 26.)
- Twin Primes? Zhang 2013: Bounded Gap (orignial: 70 million, current best: 246.)

Primes and Zeros of Zeta

Riemann 1858

- "Continuation" of $\zeta(s)$ for $s \in \mathbb{C}$.
- Euler product (product over primes) and Hadamard product (product over zeros, like a polynomial), logarithmic differentiation, ...explicit formula relating $\pi(X)$ and ζ's zeros.
- "Probably" all (nontrivial) zeros on $\mathrm{Re}(s) = \frac{1}{2}$. (RH)

Prime Number Theorem: $\pi(X) \sim X/\ln X$

- Equivalent: ζ has no zeros on $\mathrm{Re}(s) = 1$.
- Hadamard and de la Valee Poissin 1896
- "$\sim$" means that the relative error $\to 0$ as $X \to \infty$
- RH is equivalent to the optimal error term

A Spectral Approach

- Recall: Eigenvalues of a negative semidefinite symmetric matrix are $\leqslant 0$.

$$Tv \;=\; \lambda v \qquad (v \neq 0)$$

$$\lambda = -s(s-1), \qquad s = \sigma + it$$

$$\lambda \leqslant 0 \;\;\Rightarrow\;\; s \in [0,1] \text{ or } \tfrac{1}{2} + i\mathbb{R}$$

- Hilbert: find (?!) a matrix (well, operator) whose eigenvalues λ are parametrized by to zeros of zeta?

Reason to be Optimistic

Montgomery and Dyson 1972, Odlyzko 2000s

- Spacings between consecutive zeros of ζ behave, statistically, like spacings between consecutive eigenvalues of certain (large random) matrices (in the Gaussian Unitary Ensemble) whose eigenvalues correspond to energy states of large atomic nuclei. (Hamiltonians.)

Further Speculation

Perhaps the operator could be the Hamiltonian of a Quantum Chaotic system?

What is Quantum Chaos?

- Classical versus quantum system: e.g. periodic trajectories in classical billiards and eigenstates of electron.
- Classical versus chaotic behavior: e.g. classical billiards and stadium billiards.
- Quantum Chaos: What is the spectrum of a quantum system when the classical version of the system is chaotic?

Physicists would like there to be a connection between zeros of zeta and the spectrum of a QC system, because that would mean that we could find the eigenstates of the system without solving the Schrödinger equation.

Alternative: Pseudo-Laplacians

Perhaps the operator could be the Laplacian on an exotic space? e.g. "modular curve."

$$\Delta \;=\; y^2\left(\frac{\partial^2}{\partial x^2} \;=\; \frac{\partial^2}{\partial y^2}\right) \qquad \text{on} \;\; SL_2(\mathbb{Z})\backslash\mathfrak{H}$$

Eigenvalue Criterion: $\Delta u \;=\; \lambda u \;\;\Rightarrow\;\; (\Delta - \lambda)u \;=\; 0.$

- Haas 1978: purported eigenvalues λ, some of which parametrized by zeros of zetas
- Stark, Hejhal, Colin de Verdiere: Haas' λs were not really eigenvalues for Δ because Haas solved $(\Delta - \lambda)u = \delta$.
- CdV: Replace Δ with pseudo-Laplacian $\widetilde{\Delta}$ and

$$(\widetilde{\Delta} - \lambda)u \;=\; 0 \quad \overset{??}{\Longleftrightarrow} \quad (\Delta - \lambda)u = 0$$

Then genuine eigenvalues for $\widetilde{\Delta}$ would correspond to Haas' fake eigenvalues for Δ, some of which were parametrized by zeros of zeta functions.

To Construct Suitable $\widetilde{\Delta}$

- CdV tried various constructions for $\widetilde{\Delta}$ ... seemed to get something that worked ... too generally ... (implying RH for Epstein zetas, which is known to be false.)

- Recent work of Garrett and Bombieri explains subtleties
  - Why CdV did not prove RH.
  - Suggest modifications, more suitable pseudo-Laplacians ...

- Need to understand automorphic spectral theory and Sobolev theory, unbounded operators on Hilbert spaces, Friedrichs extensions, ... Lots of fun, but perhaps beyond the scope of this talk.

  Click on the following link to see more detailed notes: http://personal.stthomas.edu/dece4515/mynotes/eigvals-zeros.pdf