Spherical Functions for SL_2 from Integral Representations

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Compute by hand the (well-known) SL_2 spherical functions from their integral representations (as an alternative to solving the differential equation.)

1. Explicit integral representation

We show how to obtain an explicit integral representation for the elementary spherical functions, by left K-averaging the spherical vector in a principal series.

We model the principal series I_χ by functions on G that are left P-equivariant by a right N-invariant character χ on P. The spherical vector f_o in I_χ is right K-invariant, so is unique up to constant multiples. In fact, for $g = nak$,

$$f_o(g) \;=\; f_o(nak) \;=\; f_o(na) \;=\; \chi(na) \;=\; \chi(a) \;=\; \chi(A(g))$$

where $A(g)$ denotes the A-part of g in the NAK Iwasawa decomposition. We obtain the spherical function by averaging left K-translates of the spherical vector:

$$\varphi_\chi(g) \;=\; \int_K f_o(kg)\, dk$$

This is clearly bi-K-invariant, so, because of the Cartan decomposition $G = KA^+K$, it suffices to consider $g = a$ in A^+.

$$\varphi_\chi(a) \;=\; \int_K f_o(ka)\, dk \;=\; \int_K \chi(A(ka))\, dk$$

Recall that, using the Bruhat decomposition $G = \sqcup PwN$ for P the minimal parabolic, we may transform an integral over K to an integral over N^{op}, the opposite unipotent radical.

$$\int_{K/M} f(k)\, dk \;=\; \int_{N^{\mathrm{op}}} f(\kappa(n))\, \delta(A(n))\, dn$$

where $\kappa(g)$ denotes the Iwasawa K-part of g, and $\delta = e^{2\rho}$ is the modular function of P.

Note. This is slightly different from the formula derived in the previous document, where I used Iwasawa decomposition KAN. For computing the spherical function we need to use NAK, unless we want to model the principal series by functions that are *right* P-equivariant by a character.

Using this transformation,

$$\varphi_\chi(a) \;=\; \int_{N^{\mathrm{op}}} \chi\big(A(\kappa(n)a)\big)\, \delta(A(n))\, dn$$

Elementary Iwasawa decomposition computations enable us to re-express this integral in such a way as to avoid finding the K-part of n. (See Jorgenson-Lang, *Spherical Inversion on* $SL_n(\mathbb{R})$, IV.4.)

First we compute the A-part of gh for any $g = nak$ any h, with $kh = n'a'k'$.

$$gh \;=\; (nak)h \;=\; na(n'a'k') \;=\; nan'(a^{-1}a)a'k' \;=\; n(an'a^{-1})(aa')k'$$

Since A normalizes N, the A-part of gh is

$$A(gh) \;=\; A(g)\, A(\kappa(g)h)$$

Applying this property to $g = n$ and $h = a$,

$$A(\kappa(n)a) \;=\; A(n)^{-1}\, A(na)$$

So the integral for the spherical function can be rewritten.

$$\varphi_\chi(a) \;=\; \int_{N^{\mathrm{op}}} \chi(A(n)^{-1}\, A(na))\, \delta(A(n))\, dn \;=\; \int_{N^{\mathrm{op}}} \frac{\chi(A(na))}{\chi(A(n))}\, \delta(A(n))\, dn$$

Note. The formula in Jorgenson-Lang is slightly different, because they normalize φ_χ differently.

2. Evaluating the integrals, in the case of SL_2.

In the case of SL_2, we can compute the A-part of an arbitrary matrix by hand. Here we only need to compute the A-part of n and an.

For $SL_2(\mathbb{R})$, take

$$n_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We can solve for η in terms of x, and compute the Iwasawa A-part of n. Similarly, we compute the A-part of na:

$$\begin{aligned} A(n_x) &= \begin{pmatrix} 1/\sqrt{x^2+1} & \\ & \sqrt{x^2+1} \end{pmatrix} \\ A(n_x a_y) &= \begin{pmatrix} y/\sqrt{x^2y^4+1} & \\ & y^{-1}\sqrt{x^2y^4+1} \end{pmatrix} \end{aligned}$$

Parametrize χ by $s \in \mathbb{C}$ and evaluate χ and δ.

$$\chi(a_\eta) = e^{(\rho+i\lambda)(\log a_\eta)} = \eta^{2s+1} \quad \text{and} \quad \delta(a_\eta) = e^{2\rho(\log a_\eta)} = \eta^2$$

So the integral becomes

$$\begin{aligned} \varphi_\chi(a) &= \int_{N^{\mathrm{op}}} \frac{\chi(A(na))}{\chi(A(n))}\, \delta(A(n))\, dn \\ &= \int_{\mathbb{R}} \left(\frac{y/\sqrt{x^2y^4+1}}{1/\sqrt{x^2+1}} \right)^{2s+1} \frac{1}{x^2+1}\, dx \\ &= y^{2s+1} \int_{\mathbb{R}} \left(\frac{\sqrt{x^2+1}}{\sqrt{x^2y^4+1}} \right)^{2s+1} \frac{1}{x^2+1}\, dx \end{aligned}$$

This integral is not elementary. (It should be a K-Bessel function.)

Now consider the case of $SL_2(\mathbb{C})$.

$$n_z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$$

Solve for the Iwasawa A-parts of n and na.

$$\begin{aligned} A(n_z) &= \begin{pmatrix} 1/\sqrt{|z|^2+1} & \\ & \sqrt{|z|^2+1} \end{pmatrix} \\ A(n_z a_y) &= \begin{pmatrix} y/\sqrt{y^4|z|^2+1} & \\ & y^{-1}\sqrt{y^4|z|^2+1} \end{pmatrix} \end{aligned}$$

Parametrize χ by $s \in \mathbb{C}$ and evaluate χ and δ.

$$\chi(a_\eta) = e^{(\rho+i\lambda)(\log a_\eta)} = \eta^{2s+2} \quad \text{and} \quad \delta(a_\eta) = e^{2\rho(\log a_\eta)} = \eta^4$$

So the spherical function is

$$\begin{aligned} \varphi_\chi(a) &= \int_{\mathbb{C}} \left(\frac{y^2/(y^4|z|^2+1)}{1/(|z|^2+1)} \right)^{s+1} \frac{1}{(|z|^2+1)^2}\, dz \\ &= y^{2s+2} \int_{\mathbb{C}} \left(\frac{|z|^2+1}{y^4|z|^2+1} \right)^{s+1} \frac{1}{(|z|^2+1)^2}\, dz \\ &= 2\pi\, y^{2s+2} \int_0^\infty \left(\frac{u^2+1}{y^4u^2+1} \right)^{s+1} \frac{u}{(u^2+1)^2}\, du \\ &= 2\pi\, y^{2s+2} \int_0^\infty \left(\frac{u^2+1}{y^4u^2+1} \right)^{s-1} \frac{u}{(y^4u^2+1)^2}\, du \end{aligned}$$

For the moment, we ignore the factors out front and just consider the integral. Let $A = y^4$.

$$I \;=\; \int_0^\infty \left(\frac{u^2+1}{Au^2+1}\right)^{s-1} \frac{u}{(Au^2+1)^2}\, du$$

Make a substitution

$$v = \frac{u^2+1}{Au^2+1} \qquad\qquad dv = 2(1-A)\cdot\frac{u}{(Au^2+1)^2}\, du$$

and use the identity

$$\frac{\Gamma(s)}{z^s} \;=\; \int_0^\infty t^s e^{-tz}\, \tfrac{dt}{t}$$

The integral becomes

$$\begin{aligned}
I &= \frac{1}{2(1-A)} \int_1^{1/A} v^{s-1}\, dv \\
&= \frac{1}{2(1-A)} \int_1^{1/A} \frac{1}{\Gamma(1-s)} \int_0^\infty t^{1-s}\, e^{-tv}\, \tfrac{dt}{t}\, dv \\
&= \frac{1}{2(1-A)}\cdot\frac{1}{\Gamma(1-s)} \int_0^\infty t^{1-s} \int_1^{1/A} e^{-tv}\, dv\, \tfrac{dt}{t} \\
&= \frac{1}{2(1-A)}\cdot\frac{1}{\Gamma(1-s)} \int_0^\infty t^{1-s}\, \left(\tfrac{1}{t}e^{-tv} - \tfrac{1}{t}e^{-t/A}\right) \tfrac{dt}{t} \\
&= \frac{1}{2(1-A)}\cdot\frac{1}{\Gamma(1-s)} \left(\int_0^\infty t^{-s}e^{-tv}\, \tfrac{dt}{t} - \int_0^\infty t^{-s}e^{-t/A}\, \tfrac{dt}{t}\right)
\end{aligned}$$

Again using the identity with the Gamma function, this is

$$\begin{aligned}
I &= \frac{1}{2(1-A)}\cdot\frac{1}{\Gamma(1-s)} \left(\Gamma(-s) - \frac{\Gamma(-s)}{(1/A)^{-s}}\right) \\
&= \frac{1}{2(1-A)}\cdot\frac{\Gamma(-s)}{\Gamma(1-s)}\, (1 - A^{-s}) \\
&= \frac{1-A^{-s}}{2s(A-1)}
\end{aligned}$$

Restoring the factors out front, and recalling that $A = y^4$, we can see that the spherical function is

$$\begin{aligned}
\varphi_\chi(a_y) &= 2\pi\cdot y^{2s+2}\cdot\frac{1-y^{-4s}}{2s(y^4-1)} \\
&= \frac{\pi}{s}\cdot y^{2s+2}\cdot\frac{1-y^{-4s}}{y^4-1} \\
&= \frac{\pi}{s}\cdot y^{2s+2}\cdot\frac{y^{-2s}(y^{2s}-y^{-2s})}{y^2(y^2-y^{-2})} \\
&= \frac{\pi}{s}\cdot\frac{y^{2s}-y^{-2s}}{y^2-y^{-2}} \\
&= \frac{\pi}{s}\cdot\frac{\chi(a_y)-\chi^{-1}(a_y)}{\delta^{1/2}(a_y)-\delta^{-1/2}(a_y)}
\end{aligned}$$

If we let $y = e^{r/2}$, so that r is the Cartan radius, this is

$$\varphi_s(r) \;=\; \frac{\pi}{s}\cdot\frac{\sinh(sr)}{\sinh(r)}$$