Unbounded Operators on Hilbert Spaces

A. DeCelles
Notes on unbounded operators on Hilbert spaces
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These expository notes were written in an effort to understand some results in an appendix of a paper of Rudnick and Ueberschär [4] on eigenvalues of the point scatterer on the torus. For the most part, we follow the exposition in Garrett's vignettes [1, 2]. See also Reed-Simon [3].

Let V be a Hilbert space and D a subspace. A linear map $T : D \to V$ called an *unbounded operator on* V. This terminology is misleading since T is not necessarily defined on all of V and T may or may not be bounded. We denote the domain, D, of T as $\mathrm{Dom}(T)$. Specifying the domain is an essential part of defining an unbounded operator. Often, but not always, the domain D is chosen to be dense in V. Usually, T is not continuous when D is given the subspace topology from V.

An unbounded operator on V is *closed* (or *graph-closed*) if its graph is closed in $V \oplus V$. An unbounded operator T is *closable* if there is an operator, the *closure* $\overline{T}$ of T, whose graph is the closure of the graph of T.

For everywhere-defined linear operators the notion of closedness coincides with that of continuity: a continuous linear operator has a closed graph, and, by the Closed Graph Theorem, an everywhere-defined linear operator with a closed graph is continuous. In contrast, unbounded operators with closed graphs are *not* necessarily continous.

An unbounded operator T_2 is an *extension of* an unbounded operator T_1 if it has a larger domain and it agrees with T_1 on the domain of T_1, i.e.

$$\mathrm{Dom}(T_2) \supset \mathrm{Dom}(T_1) \qquad \text{and} \qquad T_2|_{\mathrm{Dom}(T_1)} = T_1$$

The expression $T_2 \supset T_1$ denotes that T_2 is an extension of T_1.

1 Adjoints of unbounded operators

An unbounded operator T', D' is a *subadjoint* to T, D, when

$$\langle Tv, w\rangle = \langle v, T'w\rangle \qquad \text{for all } v \in D,\ w \in D'$$

The unique maximal element among all subadjoints is the *adjoint* T^* of T; the uniqueness and existence of the adjoint requires proof.

Proposition. Let T, D be an unbounded, densely defined operator on a Hilbert space V.

1. There is a unique maximal T^*, D^* among all subadjoints to T, D.
2. T^* is closed, in the sense that its graph is closed in $V \oplus V$.
3. T^* is characterized by its graph:

$$\mathrm{Graph}(T^*) = \big(U(\mathrm{Graph}(T))\big)^{\perp}$$

where U is the Hilbert space isomorphism $V \oplus V \to V \oplus V$ given by $U(v \oplus w) = -w \oplus v$.

Proof. Note that $V \oplus V$ is a Hilbert space with inner product $\langle v \oplus w, v' \oplus w' \rangle_{V \oplus V} = \langle v, v' \rangle_V + \langle w, w' \rangle_V$. When the Hilbert space is clear from context, we drop the subscripts from $\langle \, , \, \rangle$.

We observe that the adjointness condition can be rewritten as an orthogonality condition, in the following way. For a given $w \in V$, the condition that

$$\langle Tv, w \rangle = \langle v, T'w \rangle \qquad \text{for all } v \in D$$

is equivalent to

$$0 = -\langle Tv, w \rangle + \langle v, T'w \rangle \qquad \text{for all } v \in D$$

Rewriting the right side,

$$-\langle Tv, w \rangle + \langle v, T'w \rangle = \langle -Tv, w \rangle + \langle v, T'w \rangle = \langle (-Tv) \oplus v, w \oplus T'w \rangle = \langle U(v \oplus Tv), w \oplus T'w \rangle$$

This shows that, for a given $w \in V$,

$$\langle Tv, w \rangle = \langle v, T'w \rangle \quad \text{for all } v \in D \qquad \Longleftrightarrow \qquad w \oplus T'w \in \big(U(\mathrm{Graph}(T))\big)^\perp$$

Thus an operator T', D' is subadjoint to T, D if and only if its graph lies in $\big(U(\mathrm{Graph}(T))\big)^\perp$. Constructing an operator whose graph equals $\big(U(\mathrm{Graph}(T))\big)^\perp$ proves the existence and uniqueness of the adjoint, as follows.

To a given $w \in V$, we wish to associate $w' \in V$ such that $w \oplus w' \in \big(U(\mathrm{Graph}(T))\big)^\perp$, i.e. such that $\langle Tv, w \rangle = \langle v, w' \rangle$ for all $v \in D$. This may not always be possible; but the vectors w for which is *is* possible constitute the domain of the operator we construct. Since T is densely defined, we can be sure that there is *at most* one such w', for

$$\langle Tv, w \rangle = \langle v, w_1' \rangle = \langle v, w_2' \rangle \quad \text{for all } v \in D \ \text{ (dense in } V) \qquad \Longrightarrow \qquad w_1' = w_2'$$

Thus we have a well-defined map $w \mapsto w'$ on a subset of V. This map is certainly linear, for, if $w_1 \mapsto w_1'$ and $w_2 \mapsto w_2'$ and $\alpha, \beta \in \mathbb{C}$, then, on one hand,

$$\alpha \langle Tv, w_1 \rangle + \beta \langle Tv, w_2 \rangle = \langle Tv, \alpha w_1 + \beta w_2 \rangle$$

and, on the other hand,

$$\alpha \langle Tv, w_1 \rangle + \beta \langle Tv, w_2 \rangle = \alpha \langle v, w_1' \rangle + \beta \langle v, w_2' \rangle = \langle v, \alpha w_1' + \beta w_2' \rangle$$

so $\alpha w_1 + \beta w_2 \mapsto \alpha w_1' + \beta w_2'$. By construction, its domain is the unique maximal domain among the domains of subadjoints. Since its graph is precisely $\big(U(\mathrm{Graph}(T))\big)^\perp$, and since orthogonal complements are closed, the constructed operator is a (graph-)closed operator. □

Question. From this proof it seems that the domain of the adjoint T^* of T consists precisely of those vectors $w \in V$ for which there exists $w' \in V$ such that $\langle Tv, w \rangle = \langle v, w' \rangle$ for all $v \in \mathrm{Dom}(T)$. Is this correct?

2 Symmetric and self-adjoint unbounded operators

An unbounded operator is *symmetric* when $T \subset T^*$, i.e. when (i) the domain of its adjoint is *at least as large* as the domain of T and (ii) T and its adjoint agree on the domain of T. Note that this implies

$$\langle Tv, w \rangle = \langle v, Tw \rangle \qquad \text{for all } v, w \in \mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$$

An unbounded operator is *self-adjoint* when $T = T^*$, i.e. when (i) the domain of its adjoint, which is maximal among domains of subadjoints, is *precisely* the domain of T and (ii) T and its adjoint agree on the domain of T. Note that this implies

$$\langle Tv, w \rangle = \langle v, Tw \rangle \qquad \text{for all } v, w \in \mathrm{Dom}(T) = \mathrm{Dom}(T^*)$$

For bounded operators, these two notions are the same. For symmetric unbounded operators, we wish to construct self-adjoint extensions. The Friedrichs extension is one such, as are the members of the family of extensions, parametrized by a unitary group and described in [**?**], which we discuss below.

We will need the following property of symmetric operators below.

Proposition. The eigenvalues of a symmetric operator are *real*.

Proof. Let S be a densely defined symmetric operator. Suppose $Sv = \lambda v$ for some $0 \neq v \in \mathrm{Dom}(S)$. Then, on one hand,

$$\langle Sv, v\rangle \;=\; \langle \lambda v, v\rangle \;=\; \lambda\langle v, v\rangle$$

and, on the other hand,

$$\langle Sv, v\rangle \;=\; \langle v, Sv\rangle \;=\; \langle v, \lambda v\rangle \;=\; \bar{\lambda}\,\langle v, v\rangle$$

Since $v \neq 0$, $\lambda = \bar{\lambda}$, i.e. λ is real. □

3 Symmetric extensions of symmetric operators

Note that the adjoint T^* of a densely defined symmetric operator T is a closed extension of T. However, T^* need not be symmetric. For T^* to be symmetric ($T^* \subset T^{**}$) it would have to be self-adjoint ($T^* = T^{**}$), since, in general, $T^{**} \subset T^*$ when T is symmetric, as shown in the following proposition.

Proposition. Let T be a densely defined symmetric operator. Then T is closable with $\overline{T} = T^{**}$. Further, the closure $\overline{T}$ is symmetric, $T \subset \overline{T} \subset T^*$, and $\overline{T}^{\,*} = T^*$.

Proof. Recalling that the graph of an adjoint S^* of a linear operator is

$$\mathrm{Graph}(S^*) \;=\; \big(U(\mathrm{Graph}\, S)\big)^\perp$$

where $U : V \otimes V \to V \otimes V$ is given by $v \otimes w \mapsto -w \otimes v$, and using the fact that, for any subspace X of $V \oplus V$, $\big(U(X)\big)^\perp \;=\; U^{-1}\big(X^\perp\big)$, we can see that the graph of T^{**} is the closure of the graph of T:

$$\mathrm{Graph}(T^{**}) \;=\; \big(U(\mathrm{Graph}\, T^*)\big)^\perp \;=\; \Big(U\big(\big(U(\mathrm{Graph}\, T)\big)^\perp\big)\Big)^\perp \;=\; (\mathrm{Graph}\, T)^{\perp\perp} \;=\; \overline{(\mathrm{Graph}\, T)}$$

Thus $T^{**} = \overline{T}$, and T is closable.

Since the closure of an operator can be characterized as the minimal closed extension and since T^* is a closed extension of T, we have $T \subset \overline{T} \subset T^*$.

We can see that $\overline{T}$ is symmetric, since, for all $v, w \in \mathrm{Dom}(\overline{T}) \subset \mathrm{Dom}(T^*)$,

$$\langle \overline{T}v, w\rangle \;=\; \langle T^{**}v, w\rangle \;=\; \langle v, T^*w\rangle \;=\; \langle v, \overline{T}w\rangle$$

To see that the adjoint $\overline{T}^{\,*}$ of the closure is the same as the adjoint of T, we look at the graph of $\overline{T}^{\,*} = T^{***}$. By the argument above, the graph of $(T^*)^{**}$ is the closure of the graph of T^*. Since the graph of T^* is closed, this means that the graph of T^{***} is the graph of T^*. To summarize:

$$\mathrm{Graph}(\overline{T}^*) \;=\; \overline{(\mathrm{Graph}(T^*))} \;=\; \mathrm{Graph}(T^*)$$

Thus the adjoint of $\overline{T}$ is simply T^*. □

Note. In fact, any symmetric extension S of T is a restriction of the adjoint T^*, as follows. First, $\mathrm{Dom}(S) \subset \mathrm{Dom}(T^*)$, since S is subadjoint to T. And, for all $v \in \mathrm{Dom}(T)$, $w \in \mathrm{Dom}(S)$,

$$\langle v, Sw\rangle \;=\; \langle Sv, w\rangle \;=\; \langle Tv, w\rangle \;=\; \langle v, T^*w\rangle$$

The density of $\mathrm{Dom}(T)$ implies that $T^*w = Sw$, proving that T^* agrees with S on the domain of S.

Note. Although T^{**} is a closed, symmetric extension of T, it is typically not self-adjoint. If it *is* self-adjoint, it is the *unique* self-adjoint extension of T, as follows. Suppose S is a self-adjoint extension of T. Then S is closed, and thus an extension of $\overline{T}$. It is also a restriction of T^*, since it is symmetric. Since taking adjoints is inclusion-reversing,

$$T \subset \overline{T} \subset S = S^* \subset \overline{T}^* \subset T^*$$

Self-adjointness of $\overline{T}$ means $\overline{T} = \overline{T}^*$, forcing $\overline{T} = S$.

A symmetric, densely defined operator is *essentially self-adjoint* if it has a unique self-adjoint extension. Before discussing criteria for essential self-adjointness and for the existence of self-adjoint extensions in general, we discuss one construction of self-adjoint extensions, due to Friedrichs.

4 Friedrichs' self-adjoint extension

The *Friedrichs extension* of a positive, symmetric, densely defined operator is a self-adjoint extension, which can be understood as an extension "by closure" in a certain sense. In fact, this construction works for a broader class of symmetric, densely defined operators, those that are *semi-bounded*.

A symmetric operator T is *lower semi-bounded* if there is a real constant c such that

$$\langle Tv, v\rangle \geq c \langle v, v\rangle \qquad \text{for all } v \in \mathrm{Dom}(T)$$

Positivity is a special case of lower semi-boundedness, with $c = 0$. A symmetric operator is *upper semi-bounded* if there is a real constant C such that

$$\langle Tv, v\rangle \leq C \langle v, v\rangle \qquad \text{for all } v \in \mathrm{Dom}(T)$$

Note that every semi-bounded operator can be easily obtained from a positive operator by multiplying by (-1), if necessary, and adding an appropriate constant: if T is lower semi-bounded with lower bound c, then $T - c$ is positive; if T is upper semi-bounded with upper bound C, then $C - T$ is a positive operator.

Friedrichs' construction of self-adjoint extensions is most easily described for lower semi-bounded operators with lower bound $c = 1$. We will briefly describe the Friedrichs extension, before giving the full construction in the proof of the theorem below.

Let T be a symmetric, densely defined operator on a Hilbert space V with inner product $\langle\, ,\, \rangle$. Suppose T is lower semi-bounded with lower bound $c = 1$, i.e.

$$\langle Tv, v\rangle \geq \langle v, v\rangle \qquad \text{for all } v \in \mathrm{Dom}(T)$$

We will define a *new* inner product $\langle\, ,\, \rangle_1$ on $\mathrm{Dom}(T)$ by

$$\langle v, w\rangle_1 = \langle Tv, w\rangle = \langle v, Tw\rangle \qquad \text{for all } v, w \in \mathrm{Dom}(T)$$

Let V^1 be the Hilbert space completion of $\mathrm{Dom}(T)$ with respect to the topology induced by $\langle\, ,\, \rangle_1$. Thus $\mathrm{Dom}(T)$ is a dense subspace of V^1, with respect to the $\langle\, ,\, \rangle_1$-topology, in addition to being a dense subspace of V, with respect to the $\langle\, ,\, \rangle$-topology . The $\langle\, ,\, \rangle_1$-topology is *finer* than the $\langle\, ,\, \rangle$-topology, so V^1 is a closed subspace of V.

Note. In an extended sense, $\langle\, ,\, \rangle_1$ makes sense on $V \times \mathrm{Dom}(T)$ and $\mathrm{Dom}(T) \times V$: for all $h \in V$, $v \in \mathrm{Dom}(T)$, we can define $\langle v, h\rangle_1 = \langle Tv, h\rangle$ and $\langle h, v\rangle_1 = \langle h, Tv\rangle$.

The Friedrichs extension $\widetilde{T}$ of T will be the unique self-adjoint extension of T with domain inside V^1. In particular, we will see that an element $w \in V$ is in $\mathrm{Dom}(\widetilde{T})$ precisely if there is an element $h \in V$ such that $\langle v, h\rangle = \langle v, w\rangle_1$ for all $v \in V^1$; in this case $\widetilde{T}(w) = h$. Thus we have

$$\langle \widetilde{T}v, w\rangle = \langle v, \widetilde{T}w\rangle = \langle v, w\rangle_1 \qquad \text{for all } v, w \in \mathrm{Dom}(\widetilde{T})$$

Note. In fact, asymmetric versions are also true: $\langle \widetilde{T}v, w\rangle = \langle v, w\rangle_1$ for all $v \in \mathrm{Dom}(\widetilde{T})$ and $w \in V^1$, and similarly $\langle v, \widetilde{T}w\rangle = \langle v, w\rangle_1$ for all $v \in V^1$ and $w \in \mathrm{Dom}(\widetilde{T})$.

Theorem (Friedrichs)**.** Every semi-bounded, symmetric, densely defined operator has a self-adjoint extension.

Proof. As discussed above, it suffices to consider a lower semi-bounded, symmetric, densely defined operator T, with lower bound $c = 1$. We will define an injective bounded linear operator B, whose inverse, defined on the image of B, will be the Friedrichs extension of T.

As in the discussion above, define $\langle\, ,\, \rangle_1$ on $\mathrm{Dom}(T)$ by $\langle v, w\rangle_1 = \langle Tv, w\rangle = \langle v, Tw\rangle$. We verify that $\langle\, ,\, \rangle_1$ is an inner product on $\mathrm{Dom}(T)$.

Positivity and nondegeneracy follow from the lower bound for T: $\langle v, v\rangle_1 = \langle Tv, v\rangle \geq \langle v, v\rangle \geq 0$ for all $v \in \mathrm{Dom}(T)$, and

$$0 = \langle v, v\rangle_1 = \langle Tv, v\rangle \geq \langle v, v\rangle \qquad \Longrightarrow \qquad v = 0$$

Conjugate symmetry follows from the symmetry of T and the conjugate symmetry of $\langle\, ,\, \rangle$:

$$\langle v, w\rangle_1 = \langle Tv, w\rangle = \langle v, Tw\rangle = \overline{\langle Tw, v\rangle} = \overline{\langle w, v\rangle}_1 \qquad \text{for all } v, w \in \mathrm{Dom}(T)$$

Lastly, sesquilinearity follows from the linearity of T and the sesquilinearity of $\langle\, ,\, \rangle$.

As above, let V^1 be the Hilbert space completion of $\mathrm{Dom}(T)$ with respect to the topology induced by $\langle\, ,\, \rangle_1$.

Claim. The abstract Hilbert space completion V^1 can be identified with the *closure* of $\mathrm{Dom}(T)$ with respect to the topology induced by $\langle\, ,\, \rangle_1$, we may consider it as a subspace of V.

We will define a map $B : V \to V^1$ as follows: for any $h \in V$, Bh will be the unique element of V^1 satisfying

$$\langle v, h\rangle = \langle v, Bh\rangle_1 \quad \text{or, equivalently,} \quad \langle h, v\rangle = \langle Bh, v\rangle_1 \qquad \text{for all } v \in V^1$$

That such an element exists requires proof.

Claim. The map $B : V \to V^1$ and is a well-defined, bounded linear operator. Further,

1. B is positive and symmetric (with respect to $\langle\, ,\, \rangle$) and thus self-adjoint,
2. B is injective, and
3. the image of B is dense with respect to both the topology induced by $\langle\, ,\, \rangle$ and the topology induced by $\langle\, ,\, \rangle_1$.

Proof. We start with well-definedness. For any $h \in V$, consider the linear functional $\lambda_h : v \mapsto \langle v, h\rangle$ restricted to V^1. We show that λ_h is bounded on V^1, and thus a *continous* linear functional on V^1, as follows.

$$\|\lambda_h\|_{\mathrm{op}} = \sup_{v \in V^1} \frac{|\lambda_h(v)|}{\|v\|_1} = \sup_{v \in V^1} \frac{|\langle v, h\rangle|}{\|v\|_1} \leq \sup_{v \in V^1} \frac{\|v\| \cdot \|h\|}{\|v\|_1} \leq \sup_{v \in V^1} \frac{\|v\|_1 \cdot \|h\|}{\|v\|_1} = \|h\|$$

Thus $\lambda_h \in (V^1)^*$. Since V^1 is a Hilbert space, the Riesz Representation Theorem implies that there is a unique $w \in V^1$ such that $\lambda_h(v) = \langle v, w\rangle_1$. Let $B(h) = w$. Then B is a well-defined map $V \to V^1$.

Linearity of B follows easily from the sesquilinearity of $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle_1$: for $h_1, h_2 \in V$ and $\alpha, \beta \in \mathbb{C}$,

$$\langle v, B(\alpha h_1 + \beta h_2)\rangle_1 = \langle v, \alpha h_1 + \beta h_2\rangle = \bar{\alpha}\langle v, h_1\rangle + \bar{\beta}\langle v, h_2\rangle = \bar{\alpha}\langle v, Bh_1\rangle_1 + \bar{\beta}\langle v, Bh_2\rangle_1 = \langle v, \alpha Bh_1 + \beta Bh_2\rangle_1$$

for all $v \in V^1$. Thus $B(\alpha h_1 + \beta h_2) = \alpha B(h_1) + \beta B(h_2)$.

Positivity of B follows from the positivity of $\langle\,,\,\rangle_1$: for any $h \in V$, $Bh \in V^1$, so

$$\langle Bh, h\rangle \;=\; \langle Bh, Bh\rangle_1 \;\geq\; 0$$

Symmetry of B follows from the conjugate symmetry of $\langle\,,\,\rangle_1$ and $\langle\,,\,\rangle$: for $h_1, h_2 \in V$,

$$\langle Bh_1, h_2\rangle \;=\; \langle Bh_1, Bh_2\rangle_1 \;=\; \overline{\langle Bh_2, Bh_1\rangle_1} \;=\; \overline{\langle Bh_2, h_1\rangle} \;=\; \langle h_1, Bh_2\rangle$$

Note that, since B is everywhere-defined, symmetry is the same as self-adjointness.

For injectivity, take $h_1, h_2 \in V$, and suppose $Bh_1 = Bh_2$. Then, for all $v \in V^1$,

$$\langle v, h_1\rangle \;=\; \langle v, Bh_1\rangle_1 \;=\; \langle v, Bh_2\rangle_1 \;=\; \langle v, h_2\rangle$$

Since V^1 is dense in V, this implies that $h_1 = h_2$.

Next we show that the image of B is dense in V with respect to the $\langle\,,\,\rangle$-topology. It suffices to show (?) that the kernal is trivial. Suppose $h \in \ker(B)$. Then, for all $v \in V^1$,

$$\langle v, h\rangle \;=\; \langle v, Bh\rangle_1 \;=\; \langle v, 0\rangle_1 \;=\; 0$$

Since V^1 is dense in V, this implies that $h = 0$. Since $\ker(B|_{V^1}) \subset \ker(B)$, this also implies that the image of B is dense in V^1, with respect to the $\langle\,,\,\rangle_1$-toplogy, right? …completing the proof of the claim.

Note. It is *not* true that, for a linear map $\Phi : V \to V$ on a Hilbert space V, the image is dense if and only if the kernel is trivial!

Let $A : \mathrm{img}(B) \to V$ be the inverse of B, which exists on a subspace of V. Then A is an unbounded linear operator, whose domain is dense in V^1 with respect to the $\langle\,,\,\rangle_1$-topology and dense in V with respect to the $\langle\,,\,\rangle$-topology. And, for $u \in \mathrm{Dom}(A)$, $v \in V^1$,

$$\langle Au, v\rangle \;=\; \langle u, v\rangle_1 \qquad \text{and} \qquad \langle v, Au\rangle \;=\; \langle v, u\rangle_1$$

Claim. The densely defined operator $A : \mathrm{Dom}(A) \to V$ is positive, symmetric, and in fact *self-adjoint*.

Proof. Positivity and symmetry follow from the positivity and symmetry of B. Let $v, v' \in \mathrm{Dom}(A)$. Then $v = Bh$ and $v' = Bh'$ for some $h, h' \in V$, and

$$\langle Av, v\rangle \;=\; \langle h, Bh\rangle \;=\; \langle Bh, h\rangle \;\geq\; 0$$

$$\langle Av, v'\rangle \;=\; \langle h, Bh'\rangle \;=\; \langle Bh, h'\rangle \;=\; \langle v, Av'\rangle$$

However, since A is not everywhere-defined, symmetry does not imply self-adjointness. To prove that A is self-adjoint, we will show that its graph is equal to the graph of its transpose. Since the adjoint operator is characterized by its graph, this will be sufficient to prove that A is self-adjoint.

Since A is densely defined, it has a well defined adjoint, characterized by its graph:

$$\mathrm{Graph}(A^*) \;=\; \big(U(\mathrm{Graph}A)\big)^{\perp}$$

where $U : V \oplus V \to V \oplus V$ be given by $v \oplus w \mapsto -w \oplus v$.

The self-adjointness of B implies that $\mathrm{Graph}(B) \;=\; \mathrm{Graph}(B^*) \;=\; \big(U(\mathrm{Graph}B)\big)^{\perp}$.

To relate the graph of A^* to the graph of B, we define $S : V \oplus V \to V \oplus V$ by $v \oplus w \mapsto w \oplus v$. Then clearly S interchanges the graphs of A and B. Further, $U \circ S = -S \circ U$, since $-v \oplus w = -(v \oplus -w)$, and, for any subspace X of $V \oplus V$, and $\big(S(X)\big)^{\perp} = S(X^{\perp})$ since $v \oplus w \in \big(S(X)\big)^{\perp}$ means

$$\langle v \oplus y, w \oplus x\rangle \;=\; 0 \qquad \text{for all } x \oplus y \in X$$

and $v \oplus w \in S(X^\perp)$ means

$$\langle w \oplus x, v \oplus y \rangle \;=\; 0 \qquad \text{for all } x \oplus y \in X$$

which is clearly equivalent. Thus

$$\text{Graph}(A^*) \;=\; \big(U(\text{Graph}\, A)\big)^\perp \;=\; \big(U \circ S(\text{Graph} B)\big)^\perp \;=\; \big(-S \circ U(\text{Graph}\, B)\big)^\perp \;=\; -S\big(U(\text{Graph}\, B)^\perp\big)$$

$$=\; -S\big(\text{Graph}(B^*)\big) \;=\; -S(\text{Graph}\, B) \;=\; -S(\text{Graph}\, B) \;=\; -\text{Graph}\, A \;=\; \text{Graph}\, A$$

This completes the proof of the claim.

It only remains to show A is an extension of T. We know that $\text{Dom}(A)$ and $\text{Dom}(T)$ are both subspaces of V^1, but we want to show that $\text{Dom}(A) \supset \text{Dom}(T)$ and that A agrees with T on $\text{Dom}(T)$.

Recall that the domain of A is the image of B, which consists of all $w \in V^1$ such that there is an $h \in V$ satisfying $\langle v, w\rangle_1 \;=\; \langle v, h\rangle$, for all $v \in V^1$. Clearly, for $w \in \text{Dom}(T)$, taking $h = Tw$ will work, since, by the definition of $\langle\, , \rangle_1$, $\langle v, w\rangle_1 \;=\; \langle v, Tw\rangle$ for all $v \in V^1$.

Now, since $w \in \text{Dom}(A)$, it is also true that $\langle v, w\rangle_1 \;=\; \langle v, Aw\rangle$ for all $v \in V^1$. Thus $\langle v, Tw\rangle = \langle v, Aw\rangle$ for all $v \in V^1$, and, by the density of V^1, $Aw = Tw$. Thus A is an extension of T. □

Note. The Friedrichs extension is the *unique* self-adjoint extension whose domain is contained in V^1.

Corollary. Every positive, symmetric, densely defined operator S has a Friedrichs extension $\widetilde{S}$, the unique self-adjoint extension of S, whose domain is contained in the subspace V^1, the Hilbert space completion of $\text{Dom}(S)$ with respect to the norm induced by the inner product

$$\langle v, w\rangle_1 \;=\; \langle (S+1)v, w\rangle \;=\; \langle Sv, w\rangle \,+\, \langle v, w\rangle$$

Proof. Suppose S is a positive, symmetric, densely defined operator. Let $T = S + 1$. Then Friedrichs' construction, given in the proof above, gives a self-adjoint extension $\widetilde{T}$ of T. Thus $\widetilde{S} = \widetilde{T} - 1$ is a self-adjoint extension of S. □

5 Gelfand triples and another construction of the Friedrichs extension

As above, let T be a symmetric, densely defined operator on a Hilbert space V with inner product $\langle\, , \rangle$. Suppose T is lower semi-bounded with lower bound $c = 1$, i.e.

$$\langle Tv, v\rangle \;\geq\; \langle v, v\rangle \qquad \text{for all } v \in \text{Dom}(T)$$

and let V^1 be the Hilbert space completion of $\text{Dom}(T)$ with respect to the topology induced by $\langle\, , \rangle_1$, defined on $\text{Dom}(T)$ by:

$$\langle v, w\rangle_1 \;=\; \langle Tv, w\rangle \;=\; \langle v, Tw\rangle \qquad \text{for all } v, w \in \text{Dom}(T)$$

With $j : V^1 \hookrightarrow V$ the inclusion, $j^* : V^* \to (V^1)^*$ the adjoint to this inclusion, namely $j^* : \lambda \mapsto \lambda|_{V^1}$, and $\Lambda : V \to V^*$ the Riesz-Fisher isomorphism $\Lambda : v \mapsto (u \mapsto \langle v, u\rangle)$, we have

$$V^1 \;\overset{j}{\hookrightarrow}\; V \;\underset{\approx}{\overset{\Lambda}{\longrightarrow}}\; V^* \;\overset{j^*}{\longrightarrow}\; (V^1)^*$$

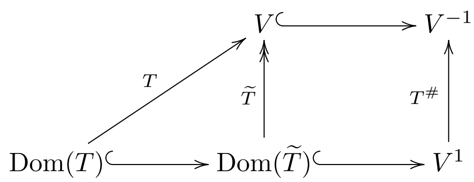
Note that j^* is injective, since $\lambda, \mu \in V^*$ with $j^*(\lambda) = j^*(\mu)$ means that λ and μ are continuous linear functionals on V that agree on the dense subspace V^1 of V, implying that $\lambda = \mu$ on V, i.e. $\lambda = \mu$ as elements of V^*. Thus, denoting $(V^1)^*$ by V^{-1}, identifying V with V^* under the isomorphism Λ on one hand and identifying V^* with its preimage under j^* on the other hand, we consider these spaces as nested: $V^1 \subset V \subset V^{-1}$ and call them a *Gelfand triple.*

Note. The inclusion of V^1 to V^{-1} is via the composite map $j^* \circ \Lambda \circ j$ rather than the canonical inclusion of a Hilbert space to its dual. In particular, this means that we consider an element $w \in V^1$ as an element of V^{-1} via $u \mapsto \langle u, w \rangle$ rather than $u \mapsto \langle u, w \rangle_1$.

The Friedrichs extension of T will be a restriction of the map $T^\# : V^1 \to V^{-1}$ given by

$$(T^\# v)(w) \;=\; \langle v, w \rangle_1 \qquad \text{for } v, w \in V^1$$

We claim that $T^\#$ agrees with T on $\mathrm{Dom}(T)$ and restricting $T^\#$ to the preimage of V under $T^\#$ gives the Fredrichs extension of T, as is expressed in the commutativity of the following diagram:



To see that $T^\#$ agrees with T on $\mathrm{Dom}(T)$, take $v \in \mathrm{Dom}(T)$, and note that we consider Tv as an element of V^{-1} by identifying Tv with the map $u \mapsto \langle u, Tv \rangle$, where u lies in V^1. Then

$$(Tv)(u) = \langle u, Tv \rangle = \langle u, v \rangle_1 = (T^\# v)(u) \qquad \text{for all } u \in V^1$$

Recall from above that the domain of the Friedrichs extension of T is

$$\mathrm{Dom}(\widetilde{T}) = \{ w \in V^1 : \text{ there is } v \in V \text{ such that } \langle u, w \rangle_1 = \langle u, v \rangle \text{ for all } u \in V^1 \}$$

This is precisely the preimage of V under $T^\#$, since the condition that $T^\# w \in V$ means there exists $v \in V$ such that $T^\# w = v$ in V^{-1}, i.e. such that the maps $u \mapsto \langle u, w \rangle_1$ and $u \mapsto \langle u, v \rangle$ agree on V^1. Certainly $\widetilde{T}$ agrees with $T^\#$ on $\mathrm{Dom}(\widetilde{T})$, since, for all $v \in \mathrm{Dom}(\widetilde{T})$,

$$(\widetilde{T}v)(u) = \langle u, \widetilde{T}v \rangle = \langle u, v \rangle_1 = (T^\# v)(u) \qquad \text{for all } u \in V^1$$

considering $\widetilde{T}v$ as an element of V^{-1} by identifying it with the map $u \mapsto \langle u, \widetilde{T}v \rangle$.

6 Criteria for the existence of self-adjoint extensions

Lemma. Suppose T is a *closed*, symmetric, densely defined unbounded operator. Let $\lambda \in \mathbb{C} - \mathbb{R}$. Then the image $(T - \lambda)\mathrm{Dom}(T)$ is closed.

Proof. (Outline) Consider a Cauchy sequence $(T - \lambda)v_i$in the image. To show that this sequence converges we will use an auxilliary operator U, which is defined as follows. Since T is symmetric, $\lambda \in \mathbb{C} - \mathbb{R}$ is not an eigenvalue for T and $(T - \lambda)$ is injective on $\mathrm{Dom}(T)$. Thus we may define an operator $U = (T - \bar{\lambda})(T - \lambda)^{-1}$ on the image $(T - \lambda)\mathrm{Dom}(T)$. This operator is unitary in the sense that $\langle Uv, Uw \rangle = \langle v, w \rangle$ for all $v, w \in (T - \lambda)\mathrm{Dom}(T)$. (See the proof of Claim 2.0.1 in [2].) The unitarity of U is used to show that $U(T - \lambda)v_i) = (T - \bar{\lambda})v_i$ is Cauchy. See the first part of the proof of Theorem 2.0.2 in [2] for the rest of the proof. □

Lemma (Claim 2.0.6 in [2])**.** Suppose T is a symmetric, densely defined unbounded operator. Let $\lambda \in \mathbb{C} - \mathbb{R}$. Then the image $(T - \lambda)\mathrm{Dom}(T)$ is dense if and only if $\bar{\lambda}$ is *not* an eigenvalue for T^*.

Proof. First we suppose the image $(T - \lambda)\mathrm{Dom}(T)$ is dense and show that $\bar{\lambda}$ is not an eigenvalue for T^*. Suppose v satisfies $(T^* - \bar{\lambda})v = 0$. Then, for all $w \in \mathrm{Dom}(T)$,

$$0 \;=\; \langle (T^* - \bar{\lambda})v, w \rangle \;=\; \langle v, (T - \lambda)w \rangle$$

Since $(T - \lambda)\mathrm{Dom}(T)$ is dense, this implies that $v = 0$, i.e. $\bar{\lambda}$ is not an eigenvalue for T^*.

Next we suppose that the image $(T-\lambda)\mathrm{Dom}(T)$ is *not* dense and show that $\bar{\lambda}$ *is* an eigenvalue for T^*. In this case, there is a nonzero vector v that is in the orthogonal complement to the image $(T-\lambda)\mathrm{Dom}(T)$. Thus, for all $w \in \mathrm{Dom}(T)$

$$0 \;=\; \langle v, (T-\lambda)w\rangle \;=\; \langle v, Tw\rangle - \langle v, \lambda w\rangle \;=\; \langle v, Tw\rangle - \langle \bar{\lambda} v, w\rangle$$

i.e. $\langle v, Tw\rangle = \langle \bar{\lambda} v, w\rangle$ for all $w \in \mathrm{Dom}(T)$. By the definition of the adjoint as the maximal subadjoint, this means that $v \in \mathrm{Dom}(T^*)$ and $T^* v = \bar{\lambda} v$. Since $v \neq 0$, this proves that $\bar{\lambda}$ is an eigenvalue for T^*. □

Theorem (von Neumann)**.** Suppose T is a closed, symmetric, densely defined unbounded operator. Let $\lambda \in \mathbb{C} - \mathbb{R}$ such that the images $(T-\lambda)\mathrm{Dom}(T)$ and $(T-\bar{\lambda})\mathrm{Dom}(T)$ are dense. Then T is self-adjoint.

Proof. It suffices to show that $\mathrm{Dom}(T^*) \subset \mathrm{Dom}(D)$. Take any $v \in \mathrm{Dom}(T^*)$, and consider $(T^* - \lambda)v$.

By the first lemma, the images $(T-\lambda)\mathrm{Dom}(T)$ and $(T-\bar{\lambda})\mathrm{Dom}(T)$ are closed. Since they are also dense, both are equal to the whole space. In particular, there is a vector $v' \in \mathrm{Dom}(T)$ such that $(T-\lambda)v' = (T^*-\lambda)v$. We will show that, in fact, $v' = v$, proving that $v \in \mathrm{Dom}(T^*)$.

For all $w \in \mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$,

$$\begin{aligned}
\langle v', (T-\bar{\lambda})w\rangle &= \langle v', (T^*-\bar{\lambda})w\rangle && \text{since } T^* \text{ is an extension of } T \\
&= \langle (T-\lambda)v', w\rangle && \text{since } v' \in \mathrm{Dom}(T^*) \text{ and } w \in \mathrm{Dom}(T) \\
&= \langle (T^*-\lambda)v, w\rangle && \text{by the definition of } v' \\
&= \langle v, (T-\bar{\lambda})w\rangle && \text{since } v \in \mathrm{Dom}(T^*) \text{ and } w \in \mathrm{Dom}(T)
\end{aligned}$$

Since $(T-\bar{\lambda})\mathrm{Dom}(T)$ is dense, this implies that $v' = v$. □

Corollary. Let T be a closed, symmetric, densely defined operator and $\lambda \in \mathbb{C} - \mathbb{R}$. If $\ker(T^*-\lambda)$ and $\ker(T^*-\bar{\lambda})$ are both trivial, then T is self-adjoint.

Proof. Since $\ker(T^*-\lambda)$ and $\ker(T^*-\bar{\lambda})$ are both trivial, neither λ nor $\bar{\lambda}$ is an eigenvalue for T^*, so by the second lemma, the images $(T-\lambda)\mathrm{Dom}(T)$ and $(T-\bar{\lambda})\mathrm{Dom}(T)$ are both dense, and, by the theorem, T is self-adjoint. □

Corollary (Criteria for essential self-adjointness)**.** Suppose T is a symmetric, densely defined unbounded operator. Let $\lambda \in \mathbb{C} - \mathbb{R}$ satisfy either one of the following conditions:

1. The images $(T-\lambda)\mathrm{Dom}(B)$ and $(T-\bar{\lambda})\mathrm{Dom}(B)$ are dense.
2. Neither λ nor $\bar{\lambda}$ are eigenvalues for the adjoint T^*.

Then T is *essentially self-adjoint*, and the closure $\overline{T}$ of T is the unique self-adjoint extension of T.

Proof. We have shown above that the two conditions are equivalent, so it suffices to show that the first condition is a criterion for essential self-adjointness. Since T is symmetric, $T \subset \overline{T} = T^{**} \subset T^*$. Thus, for all $v, w \in \mathrm{Dom}(\overline{T})$,

$$\langle \overline{T} v, w\rangle \;=\; \langle (T^*)^* v, w\rangle \;=\; \langle v, T^* w\rangle \;=\; \langle v, \overline{T} w\rangle$$

i.e. $\overline{T}$ is symmetric. Further, for any $\lambda \in \mathbb{C} - \mathbb{R}$, $(\overline{T}-\lambda)\mathrm{Dom}(\overline{T}) \supset (T-\lambda)\mathrm{Dom}(T)$, so is dense. By the theorem $\overline{T}$ is self-adjoint. □

7 Von Neumann's family of self-adjoint extensions

Let B be a densely defined symmetric operator on a Hilbert space V, and let B^* be its adjoint. For any $\eta \in \mathbb{C} - \mathbb{R}$, we define the *deficiency spaces* of B at η and $\bar{\eta}$ by

$$V_\eta(B) \;=\; \ker(B^* - \eta) \qquad V_{\bar{\eta}}(B) \;=\; \ker(B^* - \bar{\eta})$$

(Nontrivial deficiency spaces are the analogues of eigenspaces of operators on finite dimensional spaces.)

An alternate characterization of deficiency spaces will be useful.

Lemma. For $\eta \in \mathbb{C} - \mathbb{R}$, the deficiency space of a symmetric, densely defined, graph closed operator B at η can be characterized as the orthogonal complement of the image of $\mathrm{Dom}(B)$ under $(B - \bar{\eta})$, i.e.

$$V_\eta \;\stackrel{\mathrm{def}}{=}\; \ker(B^* - \eta) \;\;=\;\; \big((B - \bar{\eta})\mathrm{Dom}(B)\big)^\perp$$

Proof. Suppose $w \in V_\eta = \ker(B^* - \eta) \,\subset\, \mathrm{Dom}(B^*)$. Then, for all $v \in \mathrm{Dom}(B)$,

$$0 \;=\; \langle v, 0\rangle \;=\; \langle v, (B^* - \eta)w\rangle \;=\; \langle (B - \bar{\eta})v, w\rangle$$

i.e. $w \in \big(B - \bar{\eta})\mathrm{Dom}(B)\big)^\perp$.

On the other hand, suppose $w \in \big(B - \bar{\eta})\mathrm{Dom}(B)\big)^\perp$. In order to use the adjointness relation again, we first show that $w \in \mathrm{Dom}(B^*)$, as follows. For all $v \in \mathrm{Dom}(B)$,

$$0 \;=\; \langle (B - \bar{\eta})v, w\rangle \;=\; \langle Bv, w\rangle \,-\, \langle \bar{\eta}v, w\rangle \;=\; \langle Bv, w\rangle \,-\, \langle v, \eta w\rangle$$

i.e. $\langle Bv, w\rangle \,=\, \langle v, \eta w\rangle$ for all $v \in \mathrm{Dom}(B)$. This implies that $w \in \mathrm{Dom}(B^*)$, since the domain of B^* consists precisely of vectors w to which we may associate some w' such that $\langle Bv, w\rangle = \langle v, w'\rangle$ for all $v \in \mathrm{Dom}(B)$. Thus, by the adjointness relation,

$$0 \;=\; \langle (B - \bar{\eta})v, w\rangle \;=\; \langle v, (B^* - \eta)w\rangle \qquad \text{for all } v \in \mathrm{Dom}(B)$$

Since B is densely defined, this implies that $(B^* - \eta)w = 0$, i.e. $w \in \ker(B^* - \eta)$. □

Lemma. As a function of η, $\dim V_\eta(B)$ is a constant on the upper (lower) half plane.

The *deficiency indices* of B (at η) are the dimensions of its deficiency spaces V_η and $V_{\bar{\eta}}$. By Lemma A.1, we may refer simply to the the deficiency indices of B, as a well-defined pair, without reference to a specific η.

Theorem. Suppose B is a *closed*, symmetric, densely defined unbounded operator. Then B has deficiency indices both equal to zero if and only if B is self-adjoint.

Proof. If B has deficiency indices both equal to zero, then, for any $\eta \in \mathbb{C} - \mathbb{R}$, $\ker(B^* - \eta)$ and $\ker(B^* - \bar{\eta})$ are both trivial, so B is self-adjoint by the results in the previous section.

If, on the other hand, B is self-adjoint, then B^* is symmetric, so its eigenvalues are *real*, and, for any $\eta \in \mathbb{C} - \mathbb{R}$, the deficiency spaces $\ker(B^* - \eta)$ and $\ker(B^* - \bar{\eta})$ are trivial. □

Lemma. Suppose B is a closed, positive, symmetric operator with nonzero deficiency indices that are equal. Fix $\eta \in \mathbb{C} - \mathbb{R}$. Then for each unitary map $U : V_\eta(B) \to V_{\bar{\eta}}(B)$, there is a self-adjoint extension $B_U : D_U \to V$, where

$$D_U \;=\; \{f \;=\; g + h + Uh \;:\; g \in \mathrm{Dom}(B),\, h \in V_\eta(B)\}$$

and the action of B_U is the restriction of B^*, namely,

$$B_U f \;\;=\;\; Bg \;+\; \eta\, h \;+\; \bar{\eta}\, Uh$$

Conversely, every self-adjoint extension of B is of this form.

Proof. That B_U is an extension of B is clear, since $D_U \supset \mathrm{Dom}(B)$ and $B_U f = Bf$ for $f \in \mathrm{Dom}(B)$.

Next we show that B_U is symmetric, i.e. $\langle B_U f_1, f_2 \rangle = \langle f_1, B_U f_2 \rangle$ for any $f_1, f_2 \in D_U$.

Since $f_1 = g_1 + h_1 + Uh_1$ for some $g \in \mathrm{Dom}(B)$ and $h_1 \in V_\eta$,

$$\langle B_U f_1, f_2 \rangle \;=\; \langle Bg_1 \,+\, \eta h_1 \,+\, \bar\eta U h_1,\; f_2 \rangle \;=\; \langle Bg_1, f_2 \rangle \,+\, \langle \eta h_1, f_2 \rangle \,+\, \langle \bar\eta U h_1, f_2 \rangle$$

and since $f_2 = g_2 + h_2 + Uh_2$ for some $g_2 \in \mathrm{Dom}(B)$ and $h_1 \in V_\eta$,

$$\begin{aligned} \langle B_U f_1, f_2 \rangle \;&=\; \langle Bg_1, g_2 \rangle \,+\, \langle Bg_1, h_2 \rangle \,+\, \langle Bg_1, Uh_2 \rangle \\ &\quad + \langle \eta h_1, g_2 \rangle \,+\, \langle \eta h_1, h_2 \rangle \,+\, \langle \eta h_1, Uh_2 \rangle \\ &\quad + \langle \bar\eta U h_1, g_2 \rangle \,+\, \langle \bar\eta U h_1, h_2 \rangle \,+\, \langle \bar\eta U h_1, U h_2 \rangle \end{aligned}$$

Since B is symmetric and g_1 and g_2 are in the domain of B, $\langle Bg_1, g_2 \rangle = \langle g_1, Bg_2 \rangle$.

By definition, the deficiency space $V_\eta = \ker(B^* - \eta) \subset \mathrm{Dom}(B^*)$. Since $h_2 \in V_\eta$,

$$\langle Bg_1, h_2 \rangle \;=\; \langle g_1, B^* h_2 \rangle \;=\; \langle g_1, \eta h_2 \rangle$$

Similarly, since $Uh_2 \in V_{\bar\eta} \subset \mathrm{Dom}(B^*)$, $\langle Bg_1, Uh_2 \rangle \;=\; \langle g_1, \bar\eta U h_2 \rangle$. By the reverse argument, $\langle \eta h_1, g_2 \rangle \;=\; \langle h_1, Bg_2 \rangle$ and $\langle \bar\eta U h_1, g_2 \rangle \;=\; \langle U h_1, Bg_2 \rangle$. Thus,

$$\begin{aligned} \langle B_U f_1, f_2 \rangle \;&=\; \langle g_1, Bg_2 \rangle \,+\, \langle g_1, \eta h_2 \rangle \,+\, \langle g_1, \bar\eta U h_2 \rangle \\ &\quad + \langle h_1, Bg_2 \rangle \,+\, \langle \eta h_1, h_2 \rangle \,+\, \langle \eta h_1, U h_2 \rangle \\ &\quad + \langle U h_1, Bg_2 \rangle \,+\, \langle \bar\eta U h_1, h_2 \rangle \,+\, \langle \bar\eta U h_1, U h_2 \rangle \end{aligned}$$

Using the Hermitian property of $\langle\, , \,\rangle$ and the unitarity of U and then regrouping terms,

$$\begin{aligned} \langle B_U f_1, f_2 \rangle \;&=\; \langle g_1, Bg_2 \rangle \,+\, \langle g_1, \eta h_2 \rangle \,+\, \langle g_1, \bar\eta U h_2 \rangle \\ &\quad + \langle h_1, Bg_2 \rangle \,+\, \langle U h_1, \bar\eta U h_2 \rangle \,+\, \langle h_1, \bar\eta U h_2 \rangle \\ &\quad + \langle U h_1, Bg_2 \rangle \,+\, \langle U h_1, \eta h_2 \rangle \,+\, \langle h_1, \eta h_2 \rangle \\ \\ &=\; \langle g_1, Bg_2 \rangle \,+\, \langle g_1, \eta h_2 \rangle \,+\, \langle g_1, \bar\eta U h_2 \rangle \\ &\quad + \langle h_1, Bg_2 \rangle \,+\, \langle h_1, \eta h_2 \rangle \,+\, \langle h_1, \bar\eta U h_2 \rangle \\ &\quad + \langle U h_1, Bg_2 \rangle \,+\, \langle U h_1, \eta h_2 \rangle \,+\, \langle U h_1, \bar\eta U h_2 \rangle \\ \\ &=\; \langle f_1, B_U f_2 \rangle \end{aligned}$$

To complete the proof that B_U is self-adjoint we prove that its deficiency indices are both zero. The deficiency space at η is

$$\ker(B_U^* - \eta) \;=\; \big((B_U - \bar\eta)\mathrm{Dom}(B_U)\big)^\perp$$

Note that $(B_U - \bar\eta)\mathrm{Dom}(B_U)$ consists of functions of the form $(B_U - \bar\eta)(g + h + Uh)$ where $g \in \mathrm{Dom}(B)$ and $h \in V_\eta = \ker(B^* - \eta) = \big((B - \bar\eta)\mathrm{Dom}(B)\big)^\perp$, and in this case

$$(B_U - \bar\eta)(g + h + Uh) \;=\; (B - \bar\eta)g \,+\, (\eta - \bar\eta)h \,+\, (\bar\eta - \bar\eta)\, U h \;\;=\;\; (B - \bar\eta)g \,+\, (\eta - \bar\eta)\, h$$

Here $(B - \bar\eta)g$ ranges over $(B - \bar\eta)\mathrm{Dom}(B)$ and, since $\eta \notin \mathbb{R}$, $(\eta - \bar\eta) \neq 0$, so $(\eta - \bar\eta)h$ ranges over the orthogonal complement, the deficiency space of B at η, $V_\eta = \big((B - \bar\eta)\mathrm{Dom}(B)\big)^\perp$. Thus $(B_U - \bar\eta)\mathrm{Dom}(B_U)$ is dense in V, its orthogonal complement, the deficiency space of B_U at η, is zero, and the deficiency index of B_U at η is zero. The same argument shows that the deficiency space of B_U at $\bar\eta$ is zero. ☐

(*Proof of converse omitted.*)

Note. In particular, the Friedrichs extension can be described as a member of this family of extensions.

8 Spectrum and resolvents of unbounded operators

Another important result (for proof see [1]) regards the existence of *resolvents*: for any *densely defined*, *self-adjoint* operator T and for every $\lambda \in \mathbb{C} - \mathbb{R}$, the operator $R_\lambda = (T - \lambda)^{-1}$ is an *everywhere defined* linear operator. Further, if T is also *positive*, R_λ is everywhere defined whenever $\lambda \in [0, \infty)$.

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